Vertex algebras and the rich structures enjoyed by their categories of modules

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Joint work with
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arXiv:2107.05718

Braiding new ground:
Vertex algebras meet the Yang-Baxter Equation
Maynooth University, Ireland

Link to these slides

https://tinyurl.com/y24wwa5y

Quantum field theory in a nutshell

$$\boxed{\text{spacetimes}} \longrightarrow \boxed{\text{quantum field theory}} \longrightarrow \boxed{\text{correlation functions}}$$

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For 2D conformally invariant quantum field theories (Conformal Field Theories) there are two types of spacetimes.

- Complex curves (oriented 2-manifolds with a Riemannian conformal structure):
  - Yield chiral CFT on a complex curve.
  - Constructed via vertex algebras.
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- Conformal real 2-manifolds:
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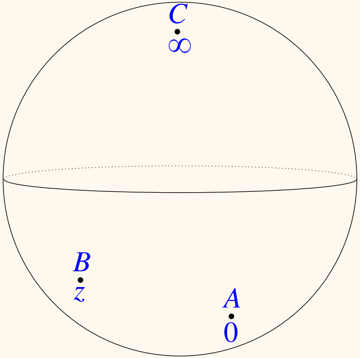
  - Yield full local CFT.
- Full local CFTs can be built by glueing together two chiral ones (holomorphic $+$ antiholomorphic $\rightarrow$ real-analytic).

Vertex algebras in a nutshell

A vertex algebra V encodes the meromorphic (i.e. single-valued) part of correlation functions on the n-pointed sphere in chiral CFT.



$$= \langle C, Y(B,z)A \rangle$$

Example with three points. $A, B, C \in V$. Conformal covariance allows us to move one point to 0 and the other to ∞.

Higher genera are obtained by gluing spheres.

Vertex algebra definition

Data:

- Vacuum module: A complex vector space V.
- Vacuum vector: A distinguished vector $\Omega \in V$.
- Field map: A linear map

$$Y : V \otimes V \to V((z)).$$

$$A \otimes B \mapsto Y(A, z)B = \sum_{n \in \mathbb{Z}} A_n B z^{-n-1}$$

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Axioms:

- State-field corespondence: $\forall A \in V$

$$Y(\Omega,z)A = A, \qquad Y(A,z)\Omega = A + zV[[z]].$$

- Locality: $\forall A, B \in$, $\exists N \gg 0$, $\forall C \in V$

$$(z-w)^N [Y(A,z), Y(B,w)]C = 0$$

Vertex algebra definition

Consequences:

- The map

$$\begin{aligned} V &\to \mathrm{End}(V)[[z, z^{-1}]] \\ A &\mapsto Y(A, z) \end{aligned}$$

  is injective.

- Translation operator: Exists linear map $T : V \to V$ such that for all $A \in V$

$$Y(TA, z) = \partial Y(A, z) = [T, Y(A, z)].$$

- Operator product expansion: For all $A, B, C \in V$

$$\left.\begin{aligned} &Y(A, z)Y(B, w)C \in V((z))((w)), \\ &Y(B, w)Y(A, z)C \in V((w))((z)), \\ &Y(Y(A, z-w)B, w)C \in V((w))((z-w)), \end{aligned}\right\}$$

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expansions of the same element in $V[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$

Conformal structures

A conformal structure is a vector $\omega \in V$ satisfying

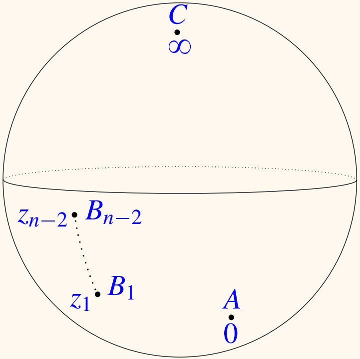
- Virasoro algebra: $Y(\omega, z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-1},\ L_m \in \mathrm{End}(V)$, with commutation relations
$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{c\cdot \mathrm{id}_V}{12}m(m^2-1), \quad c \in \mathbb{C}.$$
- $L_{-1} = T$.
- V decomposes into L_0 eigenspaces and all eigenvalues are integral.
$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \qquad V_n = \{v \in V | L_0 v = nv\}.$$

Summary vertex algebras

Vertex algebras are essentially associative commutative unital algebras with a derivation.

- Unitality comes from the state-field-correspondence property.
- Commutativity comes from the locality property.
- Associativity comes from the operator product expansion.
- Vertex algebras may have a conformal vector which yields an action of the Virasoro algebra.
- The Virasoro L_{-1}-operator becomes the derivative/translation operator.
- The Virasoro L_0-operator provides a grading (can be interpreted as energy in CFT). Usually we want the eigenspaces to be finite dimensional and the eigenvalues to be non-negative integers.

Back to the correlation function picture



$$= \langle C, Y(B_{n-2}, z_{n-2}) \cdots Y(B_1, z_1)A \rangle$$

Where $A, B_i, C \in V$ and $|z_i| > |z_{i-1}| > 0$.

Modules over a vertex algebra (V, Ω, Y)

Data

- A vector space M.
- A V-action $Y_M : V \otimes M \to M((z))$.

Axioms:

- Unit: $Y_M(\Omega, z) = \mathrm{id}_M$.
- Jacobi identity: For all $A, B \in V$, $m \in M$

$$Y_M(A,z)Y_M(B,z)m \sim Y_M(B,w)Y_M(A,z)m \sim Y_M(Y(A,z-w)B,w)m.$$

  Where $\sim$ indicates that the three series above are expansions of the same element in $M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ in different domains as in the operator product expansion property.

Modules over a vertex algebra (V, Ω, Y)

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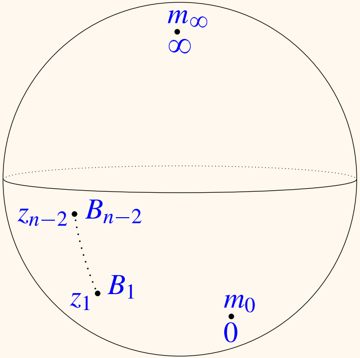
Remark

The Jacobi identity is usually expressed using formal algebraic δ-distributions which can be hard to digest when they are first encountered.

Time for another picture



$$= \langle m_\infty, Y(B_{n-2}, z_{n-2}) \cdots Y(B_1, z_1) m_0 \rangle$$

Where $B_1, \ldots, B_{n-2} \in V$, $m_0 \in M$, $m_\infty \in M^*$ and $|z_i| > |z_{i-1}| > 0$. These n-point correlators are still meromorphic.

What is M^*?

The graded dual of a module (M, Y_M) is denoted (M^*, Y_{M^*}), where

- $$M^* = \bigoplus_{\Delta \in \mathbb{C}} \hom_{\mathbb{C}}(M_\Delta, \mathbb{C}), \qquad M_\Delta = \{m \in M \ : \ (L_0 - \Delta)^N m = 0,\ N \gg 0\},$$
- and Y_{M^*} is characterised by
$$\langle Y_{M^*}(A, z)\mu, m\rangle = \langle \mu, Y_M(e^{zL_1}(-z^{-2})^{L_0} A, z^{-1})m\rangle, \qquad A \in V, m \in M, \mu \in M^*.$$

Key point: A conformal structure on V allows us to define duals of modules.

Multilinear maps

So far we have been very limited in which modules our insertion vectors can come from. To allow arbitrary insertions we need multilinear maps called intertwining operators.
Data:

- Modules M_1, M_2, M_3
- A linear map $\mathcal{Y} : M_1 \otimes M_2 \to M_3\{z\}[\log z]$. Here $\{z\}$ denotes power series in z with arbitrary complex exponents.

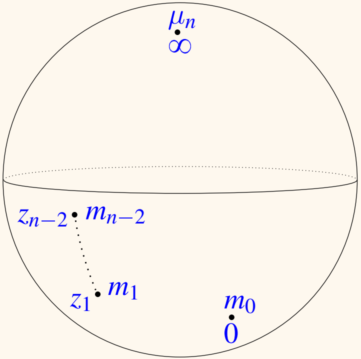
Axioms:

- Lower truncation: A lower boundedness condition for $\mathcal{Y}(m_1, z)m_2$.
- L_{-1}-derivative: $\partial_z \mathcal{Y}(m_1, z)m_2 = \mathcal{Y}(L_{-1}m_1, z)m_2$.
- Bilinearity (aka Jacobi identity):

$$Y_{M_3}(A, z)\mathcal{Y}(m_1, w)m_2 \sim \mathcal{Y}(Y_{M_1}(A, z - w)m_1, w)m_2 \sim \mathcal{Y}(m_1, w)Y_{M_2}(A, z)m_2.$$

The vector space of intertwining operators $M_1 \otimes M_2 \to M_3$ is denoted $\binom{M_3}{M_1\ M_2}$.

General correlation functions on the sphere



$$= \langle \mu_n, \mathcal{Y}(m_{n-2}, z_{n-2}) \cdots \mathcal{Y}(m_1, z_1) m_0 \rangle$$

Where $m_i \in M_i$, $i = 0, \ldots, n-1$, $\mu_n \in M_n^*$ and $|z_i| > |z_{i-1}| > 0$.

From bilinear maps to tensor products

The vertex algebra tensor product is characterised just as for vector spaces.

Let M_1, M_2 be modules over V. The V-tensor product is a module $M_1 \boxtimes M_2$ together with an intertwiner $\mathcal{Y}_{M_1,M_2} \in \binom{M_1 \boxtimes M_2}{M_1 \ \ M_2}$ such that for every other module X and intertwiner $\mathcal{X} \in \binom{X}{M_1 \ \ M_2}$ the following triangle commutes.

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "\mathcal{Y}_{M_1,M_2}"] \arrow[dr, "\mathcal{X}"'] & M_1 \boxtimes M_2\{z\}[\log(z)]. \arrow[d, dashed, "\exists! \phi"] \\ & X\{z\}[\log(z)] \end{tikzcd}$$

Alternatively, $M_1 \boxtimes M_2$ is a representing object for the functor

$$\begin{aligned} V\text{-mod} &\to \mathsf{Vec} \\ X &\mapsto \binom{X}{M_1 \ \ M_2}. \end{aligned}$$

Tensor categories from vertex algebras

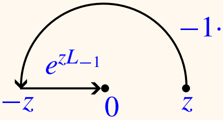
Provided V-mod is well chosen the previous universal property generates the structure of a tensor category:

- For $f_1 : M_1 \to N_1, f_2 : M_2 \to N_2$
$$\mathcal{Y}_{N_1,N_2} \circ f_1 \otimes f_2 = f_1 \boxtimes f_2 \circ \mathcal{Y}_{M_1,M_2}.$$
- Unitors (left now, right later):
$$Y_M(A,z)m = \ell_M(\mathcal{Y}_{V,M}(A,z)m)$$
- Associator:
$$\mathcal{Y}_{M_1 \boxtimes M_2, M_3}(\mathcal{Y}_{M_1,M_2}(m_1, z_1 - z_2)m_2, z_2)m_3 = A_{M_1,M_2,M_3}(\mathcal{Y}_{M_1,M_2 \boxtimes M_3}(m_1, z_1)\mathcal{Y}_{M_2,M_3}(m_2, z_2)m_3).$$

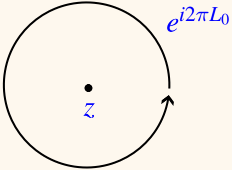
Tensor categories from vertex algebras

The conformal structure gives a balanced braiding and twist.

Braiding c



Twist θ



- $$c_{M_1,M_2}(\mathcal{Y}_{M_1,M_2}(m_1,z)m_2) = e^{zL_{-1}}\mathcal{Y}_{M_2,M_1}(m_2,e^{i\pi}z)m_1.$$
- $$\theta_M = e^{i2\pi L_0}\big|_M .$$
- $$\theta_{M_1\boxtimes M_2} = c_{M_2,M_1}\circ c_{M_1,M_2}\circ(\theta_{M_1}\boxtimes\theta_{M_2}).$$

Duality structures

Similar to how modules can be dualised, intertwining operators can be too. If $\mathcal{Y} \in \binom{M_3}{M_1\ M_2}$, then $\mathcal{Y}^*$ characterised by

$$\langle \mathcal{Y}^*(m_1,z)\mu_3, m_2\rangle = \langle \mu_3, \mathcal{Y}(e^{zL_1}(-z^{-2})^{L_0}m_1, z^{-1})m_2\rangle$$

is an intertwining operator in $\binom{M_2^*}{M_1\ M_3^*}$, i.e. there is a natural isomorphism $\binom{M_3}{M_1\ M_2} \cong \binom{M_2^*}{M_1\ M_3^*}$.

This gives us natural isomorphisms

$$\mathrm{Hom}(M,N^*) \cong \mathrm{Hom}(M\boxtimes V, N^*) \cong \binom{N^*}{M\ V} \cong \binom{V^*}{M\ N} \cong \mathrm{Hom}(M\boxtimes N, V^*).$$

Grothendieck-Verdier categories or $*$-autonomous categories

Definition: Grothendieck Verdier Category

Let $\mathcal{C}$ be a monoidal category. An object $K \in \mathcal{C}$ is called *dualising*, if the following hold.

- For each $Y \in \mathcal{C}$ the functor
$$X \mapsto \mathrm{Hom}(X \otimes Y, K)$$
is representable, that is, for each Y there exists a $DY \in \mathcal{C}$ and a natural isomorphism ϖ such that
$$\varpi_{X,Y} : \mathrm{Hom}(X \otimes Y, K) \cong \mathrm{Hom}(X, DY).$$
This defines a contravariant functor $D : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}}$.
- The functor D is an anti-equivalence.

A Grothendieck-Verdier category is a monoidal category $\mathcal{C}$ together with a choice of dualising object K.

Grothendieck-Verdier categories or $*$-autonomous categories

Remarks

- Grothendieck-Verdier structures generalise rigid duality. In a rigid category the tensor unit is dualising. However, a rigid category may admit many more dualising objects (specifically: all invertible objects). If the unit object is dualising, this does not imply that the category is rigid.
- Categories of modules over vertex algebras are naturally Grothendieck-Verdier with the graded dual V of the vertex algebra as the dualising object. As we proved two slides ago:
$$\mathrm{Hom}(M, N^*) \cong \mathrm{Hom}(M \boxtimes N, V^*)$$
The dualising functor D is just taking graded duals: $DM = M^*$.
- The existence of dualising objects implies that the tensor product admits a right adjoint and hence that the tensor product is right exact, if the category is abelian.

Grothendieck-Verdier categories or $*$-autonomous categories

Foundational literature on vertex algebra tensor categories:

- Huang 2008: The category of modules over a rational vertex algebra (c_2-cofinite and semisimple) is a modular tensor category.
- Huang, Lepowsky, Zhang: 1012.4193, 1012.4196, 1012.4197, 1012.4198, 1012.4199, 1012.4202, 1110.1929, 1110.1931
  Logarithmic tensor product theory I-VIII: The comprehensive encyclopedia on tensor products for vertex algebras.
- Creutzig, Kanade, McRae '24: An accessible review vertex algebra tensor product theory with a focus on vertex super algebras.

Grothendieck-Verdier categories or $*$-autonomous categories

Some of my contributions to results on duality/rigidity.

- Gaberdiel, Runkel, SW, '09: First example of a c_2-cofinite vertex algebra where the tensor product was only right exact but not left exact.
- Tsuchiya, SW, '13: First proof of the category of modules over a c_2-cofinite but not semisimple vertex algebra being rigid.
- Allen, SW, '22: First proof of the category of modules over a non-c_2-cofinite but not semisimple vertex algebra being rigid.
- Allen, Lentner, Schweigert, SW '24: Discovery of Grothendieck-Verdier structure on categories of vertex algebra modules.
- Nakano, Orosz-Hunziker, Ros Camacho, Wood '26: Proof of the category of modules over simple affine $\mathfrak{sl}(2)$ at admissible levels being rigid.

Summary

- Vertex algebras generalise commutative algebras (perhaps braided algebras would have been a more descriptive name for them).
- Vertex algebras can admit a conformal structure. It need not be unique.
- In analogy to commutative algebras, we have modules and tensor products so categories of modules become tensor categories.
- There is a natural braiding coming from translations (derivatives). Additionally conformal structures give a twist and a notion of duals.
- Grothendieck-Verdier categories are the natural categorical framework for categories of vertex algebra modules.

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Thank you