On quasi-lisse vertex algebras and conformal embeddings

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Braiding New Ground: Vertex Algebras meet the Yang–Baxter Equation
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Based on papers+ some ongoing work

- D. A., A. Milas; Vertex algebras related to regular representations of SL_2, Advances in Mathematics 487 (2026), 110763, arXiv:2502.01766.
- D. A., I. Vukorepa; A new quasi-lisse affine vertex algebra of type D_4, Commun. Contemp. Math. (2026), arXiv:2504.13783.
- D. A., T. Arakawa, T. Creutzig, A. Linshaw, A. Moreau, P. Möseneder Frajria, P. Papi; W–algebras as conformal extensions of affine VOAs, Advances in Mathematics (2026) arXiv:2508.18889.
- Previous papers on collapsing levels and conformal embeddings.

Motivation

- Construction and classification of new quasi-lisse vertex algebras.
- Affine vertex algebras at non-admissible levels and their representation theory.
- Application of affine W–algebras and quantum Hamiltonian reduction to the study of $L_k(\mathfrak{g})$.
- Classification of conformal and collapsing levels.
- Decomposition of conformal embeddings and applications to the category KL_k.

Vertex algebras

A vertex algebra consists of a vector space V, a vacuum vector $\mathbf{1}$, a translation operator T, and a linear map

$$Y(\cdot, z) : V \longrightarrow \mathrm{End}(V)[[z, z^{-1}]], \qquad Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

where $a_{(n)} b = 0$ for $n \gg 0$, such that

- $Y(\mathbf{1}, z) = \mathrm{id}_V$ and $Y(a, z)\mathbf{1} \in a + zV[[z]]$;
- $T\mathbf{1} = 0$ and $[T, Y(a, z)] = \partial_z Y(a, z)$;
- the vertex operators are mutually local: for every $a, b \in V$,

  $$(z - w)^N [Y(a, z), Y(b, w)] = 0$$

  for $N \gg 0$.

C_2–algebra and associated variety

Let V be a vertex algebra. Set

$$C_2(V) = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\}, \qquad R_V = V/C_2(V).$$

Then R_V is a commutative Poisson algebra with

$$\overline{a}\,\overline{b} = \overline{a_{(-1)}b}, \qquad \{\overline{a}, \overline{b}\} = \overline{a_{(0)}b}.$$

The associated variety of V is the affine Poisson variety

$$X_V = \mathrm{Specm}\, R_V.$$

Lisse and quasi-lisse vertex algebras

- A vertex algebra V is *lisse* if
$$\dim X_V = 0.$$
Thus R_V is finite-dimensional, and V is C_2–cofinite.
- A lisse vertex algebra has only finitely many irreducible modules. The triplet vertex algebra $\mathcal{W}(p)$ is lisse and has $2p$ irreducible modules.
- A vertex algebra V is *quasi-lisse* if X_V has finitely many symplectic leaves.
- A quasi-lisse vertex algebra has only finitely many irreducible ordinary modules.
- For an affine vertex algebra,
$$L_k(\mathfrak{g}) \text{ is quasi-lisse} \quad \Longleftrightarrow \quad X_{L_k(\mathfrak{g})} \subset \mathcal{N},$$
where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$.

Basic concepts

- Let $\mathfrak{g}$ be a simple Lie algebra and

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K.$$

- $V^k(\mathfrak{g})$ is the universal affine vertex algebra and $L_k(\mathfrak{g})$ its simple quotient. For $k \neq -h^\vee$,

$$c_k(\mathfrak{g}) = \frac{k \dim \mathfrak{g}}{k + h^\vee}.$$

- For a nilpotent element $f \in \mathfrak{g}$,

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0_{DS,f}\big(V^k(\mathfrak{g})\big),$$

  and $\mathcal{W}_k(\mathfrak{g}, f)$ denotes its simple quotient.

- KL_k consists of $L_k(\mathfrak{g})$–modules which are locally finite as $\mathfrak{g}$–modules and admit a decomposition into generalized $L(0)$–eigenspaces with eigenvalues bounded below.

Affine vertex algebras and conformal embeddings

Let $U \subset V$ be vertex algebras with conformal vectors ω_U and ω_V.

- The embedding $U \hookrightarrow V$ is conformal if

$$\omega_U = \omega_V.$$

- Equality of the corresponding central charges is a necessary condition.
- In this case V is a conformal extension of U.
- Main problems: classify conformal embeddings and decompose V as a U–module.

Affine W–algebras and collapsing levels

- The fields of conformal weight one generate a natural affine vertex subalgebra
$$V^{k^\natural}(\mathfrak{g}^\natural) \longrightarrow \mathcal{W}^k(\mathfrak{g}, f).$$
- The level k is conformal if the image of $V^{k^\natural}(\mathfrak{g}^\natural)$ is conformally embedded in $\mathcal{W}_k(\mathfrak{g}, f)$.
- The level k is collapsing if
$$\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural).$$
- It is totally collapsing if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}\mathbf{1}$.
- A general criterion gives an almost complete list of conformal and collapsing levels. [D.A.–Arakawa–Creutzig–Linshaw–Moreau–Möseneder Frajria–Papi]

Minimal affine W–algebras

For $f = f_\theta$, the minimal grading is

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \qquad \mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x.$$

The universal algebra $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ is strongly and freely generated by

- $J^{\{a\}}$, $a \in \mathfrak{g}^\natural$, of conformal weight one;
- $G^{\{v\}}$, $v \in \mathfrak{g}_{-1/2}$, of conformal weight $3/2$;
- the Virasoro vector L.

There is a canonical quadratic polynomial $p_\mathfrak{g}(k)$ such that

$$G^{\{u\}}_{(2)} G^{\{v\}} = 4(e_\theta \mid [u, v]) p_\mathfrak{g}(k) \mathbf{1}.$$

The central charge is

$$c(\mathfrak{g}, k, f_\theta) = \frac{k \dim \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

The minimal case

Let $f = f_\theta$ be a minimal nilpotent element.

Theorem

The affine embedding

$$L_{k^\natural}(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}_k(\mathfrak{g}, f_\theta)$$

is conformal if and only if the corresponding central charges are equal. If a conformal level is non-collapsing, then

$$k = -\frac{2}{3}h^\vee \qquad \text{or} \qquad k = -\frac{h^\vee - 1}{2}.$$

The level k is collapsing if and only if

$$p_{\mathfrak{g}}(k) = 0,$$

where $p_{\mathfrak{g}}$ is an explicit quadratic polynomial.

D.A.–Kac–Möseneder Frajria–Papi–Perše, 2017–2018.

Collapsing levels in the minimal case

The level k is collapsing if and only if $p_{\mathfrak{g}}(k) = 0$.

$\mathfrak{g}$	$p_{\mathfrak{g}}(k)$	$\mathfrak{g}$	$p_{\mathfrak{g}}(k)$
$\mathfrak{sl}(m\|n),\ m \neq n$	$(k+1)\big(k+\frac{m-n}{2}\big)$	E_6	$(k+3)(k+4)$
$\mathfrak{psl}(m\|m)$	$k(k+1)$	E_7	$(k+4)(k+6)$
$\mathfrak{osp}(m\|n)$	$(k+2)\big(k+\frac{m-n-4}{2}\big)$	E_8	$(k+6)(k+10)$
$\mathfrak{spo}(n\|m)$	$(k+\frac{1}{2})\big(k+\frac{n-m+4}{4}\big)$	F_4	$(k+\frac{5}{2})(k+3)$
$D(2,1;a)$	$(k-a)(k+1+a)$	G_2	$(k+\frac{4}{3})(k+\frac{5}{3})$
$F(4),\ \mathfrak{g}^\natural = \mathfrak{so}_7$	$(k+\frac{2}{3})(k-\frac{2}{3})$	$G(3),\ \mathfrak{g}^\natural = G_2$	$(k-\frac{1}{2})(k+\frac{3}{4})$
$F(4),\ \mathfrak{g}^\natural = D(2,1;2)$	$(k+\frac{3}{2})(k+1)$	$G(3),\ \mathfrak{g}^\natural = \mathfrak{osp}(3\|2)$	$(k+\frac{2}{3})(k+\frac{4}{3})$

The classification includes basic Lie superalgebras.

D.A.–Kac–Möseneder Frajria–Papi–Perše.

Motivating example: Cvitanović–Deligne series

- Affine vertex algebra $L_k(\mathfrak{g})$, with $k = -\frac{h^\vee}{6} - 1$, where $\mathfrak{g}$ is of type

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

- The Cvitanović–Deligne series is characterized by the fact that this is a totally collapsing level for the minimal W–algebra:

$$\mathcal{W}_k(\mathfrak{g}, f_\theta) \cong \mathbb{C}\mathbf{1}.$$

- The associated variety is $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\text{min}}}$, the closure of the minimal nilpotent orbit.
- $L_k(\mathfrak{g})$ is quasi-lisse, and we have, up to equivalence, one irreducible module in $KL_k(\mathfrak{g})$.

How to find new examples of vertex algebras with these properties?

Example: $L_{-\frac{5}{3}}(G_2)$

- There exists a conformal embedding

$$L_{-\frac{5}{3}}(\mathfrak{sl}_2) \otimes L_{-5}(\mathfrak{sl}_2) \hookrightarrow L_{-\frac{5}{3}}(G_2).$$

- $L_{-\frac{5}{3}}(G_2)$ is quasi-lisse and belongs to the Cvitanović–Deligne series.
- Problem: determine the decomposition of this conformal embedding.
- Our proof of decompostion uses a free-field realization and inverse quantum Hamiltonian reduction.

Example: $L_{-\frac{5}{3}}(G_2)$

- The complete decomposition is

$$L_{-\frac{5}{3}}(G_2) \cong \bigoplus_{\ell=0}^{\infty} L_{\widehat{\mathfrak{sl}}_2}\left(-(5+3\ell)\Lambda_0 + 3\ell\Lambda_1\right) \otimes L_{\widehat{\mathfrak{sl}}_2}\left(-(\tfrac{5}{3}+\ell)\Lambda_0 + \ell\Lambda_1\right).$$

- Thus $L_{-\frac{5}{3}}(G_2)$ is a multiplicity-free infinite conformal extension of

$$L_{-\frac{5}{3}}(\mathfrak{sl}_2) \otimes L_{-5}(\mathfrak{sl}_2).$$

Generalization: the vertex algebra $\mathcal{C}_p(\mathfrak{g})$

- Let $\mathfrak{g}$ be a simple Lie algebra. Can we construct a vertex algebra which is quasi-lisse and a conformal extension of

$$L_{-h^\vee+\frac{1}{p}}(\mathfrak{g}) \otimes L_{-h^\vee-p}(\mathfrak{g})?$$

- For $\mathfrak{g} = \mathfrak{sl}_2$, the construction is an affine analogue of the Peter–Weyl decomposition of $\mathbb{C}[SL_2]$.

Theorem (D. A., A. M. 2026)

For every $p \geq 1$, there is a simple vertex algebra $\mathcal{C}_p := \mathcal{C}_p(\mathfrak{sl}_2)$ such that

$$L_{-2+\frac{1}{p}}(\mathfrak{sl}_2) \otimes L_{-2-p}(\mathfrak{sl}_2) \hookrightarrow \mathcal{C}_p$$

is conformal, and

$$\mathcal{C}_p = \bigoplus_{\ell=0}^{\infty} L_{\widehat{\mathfrak{sl}}_2}(-(2+p+p\ell)\Lambda_0 + p\ell\Lambda_1) \otimes L_{\widehat{\mathfrak{sl}}_2}(-(2-\tfrac{1}{p}+\ell)\Lambda_0 + \ell\Lambda_1).$$

In particular, $\mathcal{C}_2 = M_{(2)}$ and $\mathcal{C}_3 = L_{-\frac{5}{3}}(G_2)$.

Is $\mathcal{C}_p$ quasi-lisse?

- We expect that $\mathcal{C}_p$ is quasi-lisse for every $p \geq 2$. This is proved for $p = 2, 3, 4, 5$.
- For every $p \geq 2$, $\mathcal{C}_p$ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$–graded with finite-dimensional graded subspaces and convergent characters. Its characters exhibit modularity.
- We prove
$$\mathcal{C}_4 \cong \mathcal{W}_{-\frac{23}{4}}(F_4, A_1 + \widetilde{A}_1).$$
- The level $-\frac{23}{4}$ is admissible and $\mathcal{C}_4$ is quasi-lisse.
- Similarly, $\mathcal{C}_5$ is quasi-lisse and
$$\mathcal{C}_5 \cong \mathcal{W}_{-30+\frac{31}{5}}(E_8, A_4 + A_2).$$

A general criterion for conformal embeddings

Let V be a vertex algebra and let

$$\varphi : V^k(\mathfrak{a}) \longrightarrow V, \qquad \varphi(\mathfrak{a}) = V_1.$$

Assume that $k + h^\vee \notin \mathbb{Q}_{\leq 0}$, that $\varphi(x)$ is primary for every $x \in \mathfrak{a}$, and that

$$c\big(V^k(\mathfrak{a})\big) = c(V), \qquad \dim \mathrm{Com}\big(\varphi(V^k(\mathfrak{a})), V\big)_2 = 1.$$

Then the simple quotient of V is a conformal extension of the image of $V^k(\mathfrak{a})$. Applied to affine W–algebras, this gives an almost complete list of conformal and collapsing levels.

D.A.–Arakawa–Creutzig–Linshaw–Moreau–Möseneder Frajria–Papi, arXiv:2508.18889v2.

Two conformal embeddings from the classification

In the joint work, we construct the conformal embeddings

$$L_{-6}(\mathfrak{sl}_2) \otimes L_{-7/4}(\mathfrak{sl}_2) \hookrightarrow \mathcal{W}_{-23/4}(F_4, A_1 + \widetilde{A}_1)$$

and

$$L_{-7}(\mathfrak{sl}_2) \otimes L_{-9/5}(\mathfrak{sl}_2) \hookrightarrow \mathcal{W}_{-30+31/5}(E_8, A_4 + A_2).$$

Problem: determine the decompositions of these conformal embeddings.

D.A.–Arakawa–Creutzig–Linshaw–Moreau–Möseneder Frajria–Papi, arXiv:2508.18889v2.

Decomposition of the two W–algebra embeddings

The identifications

$$\mathcal{C}_4 \cong \mathcal{W}_{-23/4}(F_4, A_1 + \widetilde{A}_1)$$

and

$$\mathcal{C}_5 \cong \mathcal{W}_{-30+31/5}(E_8, A_4 + A_2)$$

give the complete decompositions of the two conformal embeddings from the new classification.

The vertex algebras $\mathcal{C}_p$, $p = 2, 3, 4, 5$, are quasi-lisse.

D.A.–Milas, Adv. Math. 487 (2026), 110763; D.A.–Arakawa–Creutzig–Linshaw–Moreau–Möseneder Frajria–Papi, arXiv:2508.18889v2.

Decompositions and tensor categories

Let $U \hookrightarrow V$ be a conformal embedding and let

$$V = \bigoplus_{i \in I} M_i$$

be the decomposition of V as a U–module.

- This decomposition relates the tensor categories of U–modules and V–modules.
- It can be used to determine fusion rules and to study rigidity.

The totally collapsing level $k = -14/3$

Let $f_{\rm sreg}$ be the subregular nilpotent element of D_4, corresponding to the partition $[5,3]$. Then

$$\mathcal{W}_{-14/3}(D_4, f_{\rm sreg}) \cong \mathbb{C}, \qquad X_{L_{-14/3}(D_4)} = \overline{\mathbb{O}}_{\rm sreg}.$$

- The maximal ideal in $V^{-14/3}(D_4)$ is generated by three singular vectors of conformal weight six.
- $L_{-14/3}(D_4)$ has 405 irreducible modules in category $\mathcal{O}$, and only one irreducible module in $KL_{-14/3}(D_4)$.
- $L_{-14/3}(D_4)$ is quasi-lisse and $KL_{-14/3}(D_4)$ is semisimple.

D.A.–Vukorepa, Commun. Contemp. Math. (2026).

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Thank you