# From indecomposable solutions to the Yang–Baxter equation to irreducible representations

Joint work with C. Dietzel and E. Feingesicht
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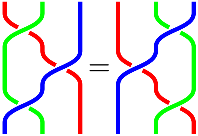
# YANG–BAXTER EQUATION



**Set-theoretic solutions of the YBE:**

A non-empty set $X$ and a **bijective** map $r : X \times X \to X \times X$ such that

$$(r\times \mathrm{id})\circ(\mathrm{id}\times r)\circ(r\times \mathrm{id}) = (\mathrm{id}\times r)\circ(r\times \mathrm{id})\circ(\mathrm{id}\times r).$$

We will use the notation $r(x,y) = (\lambda_x(y), \rho_y(x))$. A solution is:

- **non-degenerate** if $\lambda_x, \rho_x$ are bijective for all $x \in X$;
- **involutive** if $r^2 = \mathrm{id}$.

With **solution** we mean **finite involutive non-degenerate** set-theoretic solution to the Yang–Baxter equation. In this case:

$$\rho_y(x) = \lambda^{-1}_{\lambda_x(y)}(x).$$

# STRUCTURE BRACE

The **structure group** of a solution $r(x,y) = (\lambda_x(y), \rho_y(x))$ is

$$\mathbf{G(X,r)} = \langle X \mid x \circ y = \lambda_x(y) \circ \rho_y(x) \rangle.$$

- $(G(X,r), +, \circ)$ is a **brace** with $+$ induced by
$$x + y = x \circ \lambda_x^{-1}(y) \text{ for } x, y \in X.$$
- $(G(X,r), +)$ is a free abelian group on $X$.
- $\lambda$ extends to a a group homomorphism
$$\lambda : (G(X,r), \circ) \to \mathrm{Aut}(G(X,r), +)$$
- The **Dehornoy class** of $(X, r)$ is the smallest positive integer $d$ such that $dx \in \ker \lambda$ for every $x \in X$.

# MONOMIAL REPRESENTATION

Let $(X, r)$ be a solution of Dehornoy class $d$.
and we define

- $P_{\lambda_g}$ the permutation matrix associated with $\lambda_g \in Perm(X)$
- $D_x = diag(1, \ldots, 1, \underset{x^{th}\text{-index}}{q}, 1, \ldots, 1)$ in $Mat_X(\mathbb{C}(q))$

## Theorem (Dehornoy, 2015)

$$\Theta : (G(X), \circ) \to M_X(\mathbb{C}(q))$$

$$g = \sum_{x \in X} g_x x \mapsto \left( \prod_{x \in X} D_x^{g_x} \right) P_{\lambda_g}$$

*is a faithful representation. Moreover for every positive integer* $k$*, the evaluation* $q = \zeta_{kd}$ *yields a faithful representation*

$$\overline{\Theta}_k : \overline{G}_k(X, r) = G(X, r)/kdG(X, r) \to M_X(\mathbb{C}),$$

# INDECOMPOSABILITY VS IRREDUCIBILITY

Recall that $(X, r)$ is **indecomposable** if there is no non-trivial partition $X = A \sqcup B$ such that $r(A^2) = A^2$ and $r(B^2) = B^2$.

**What is the connection between indecomposability of the solution and irreducibility of the representations?**

- If $(X, r)$ is decomposable, then $\Theta$ and $\overline{\Theta_k}$ are reducible:

  if $X = A \sqcup B$ is a decomposition, then the subspaces of $\mathbb{C}(q)^X$ spanned by $A$ and $B$ are invariant.

# INDECOMPOSABILITY VS IRREDUCIBILITY - CASE $K > 1$

Let $(X, r)$ be a solution of size $n$ and Dehornoy class $d$. Recall

$$\Theta : G(X) \to M_X(\mathbb{C}(q)) \qquad \overline{\Theta}_k : \overline{G_k(X)} \to M_X(\mathbb{C})$$

## Theorem (C. Dietzel, E. Feingesicht, S.P., 2025)

*Suppose that $k > 1$, then the following are equivalent:*

1. *$X$ is indecomposable,*
2. *$\Theta$ is irreducible,*
3. *$\overline{\Theta}_k$ is irreducible.*

# INDECOMPOSABILITY VS IRREDUCIBILITY - CASE $K = 1$

**What about the case $k = 1$?**

Let $(X, r)$ be a solution of size $n$ and Dehornoy class $d$. Recall

$$\Theta : G(X) \to M_X(\mathbb{C}(q)) \qquad \overline{\Theta}_k : \overline{G_k(X)} \to M_X(\mathbb{C})$$

**Theorem (C. Dietzel, E. Feingesicht, S.P., 2025)**

*If $(X, r)$ is indecomposable, $\overline{\Theta}_1$ is irreducible if and only if*

$$d > 2 \qquad \text{or} \qquad d = 2 \text{ and } |\mathcal{G}(X, r)| < 2^{\frac{n}{2}}.$$

There is an example of an indecomposable solution wit $d = 2$ whose representation $\overline{\Theta}_1$ is reducible

# Thank you!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