Chain Conditions on Skew Braces and Solutions of the Yang-Baxter Equation

Braiding new ground: Vertex algebras meet the Yang-Baxter Equation, Maynooth, 17/9/2026

Massimiliano Di Matteo

(joint work with Ramon Esteban-Romero, Maria Ferrara and Vicent Pérez-Calabuig)

Università degli Studi della Campania “Luigi Vanvitelli”

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Definition

Let $(B, +, \circ)$ be a skew brace, then

- B is called **s-noetherian** (resp. **s-artinian**) if it satisfies the maximal (resp. minimal) condition on sub skew braces;
- B is called **i-noetherian** (resp. **i-artinian**) if it satisfies the maximal (resp. minimal) condition on ideals.

M. Di Matteo, R. Esteban Romero, M. Ferrara and V. Pérez-Calabuig, "Chain Conditions on Skew Braces and Solutions of the Yang–Baxter Equation", arXiv:2606.28177v2

Soluble Skew Braces

Hall Theorem, 1954

A soluble group satisfying max-n is finitely generated.

P. Hall, "Finiteness Conditions for Soluble Groups", *Proc. Lond. Math. Soc.* 3 (1954), No. 1 419–436

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Theorem

Let $(B, +, \circ)$ be an i-noetherian soluble skew brace, which is either two-sided or have $(B, +)$ abelian, then it is finitely generated.

A. Ballester-Bolinches, R. Esteban-Romero, P. Jiménez-Seral, and V. Pérez-Calabuig, “Soluble Skew Left Braces and Soluble Solutions of the Yang–Baxter Equation”, *Adv. Math.* 455 (2024), 109880, 27pp.

McLain Theorem, 1956

Let G be a locally nilpotent group, then the following are equivalent:

- G satisfies max-n;
- G satisfies max;
- G is a finitely generated group.

DH. McLain, "On Locally Nilpotent Groups", *Math. Proc. Camb. Ph. Soc.* 52 (1956), No. 1 5–11

Locally multipermutational solutions

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Theorem

An i-noetherian locally multipermutational skew brace is finitely generated.

Counterexample

$$(G,+) = \bigoplus_{n\in\mathbb{Z}} (\mathbb{Z},+),$$

$\mathcal{B} = G \times \langle x \rangle$, where $\langle x \rangle$ is an infinite cyclic group, and

$$\lambda : \langle x \rangle \longrightarrow \mathrm{Aut}(G,+),$$

such that $\lambda_x(1_n) = 1_{n+1}$ for any $n \in \mathbb{Z}$.

It is possible to structure $\mathcal{B}$ as a skew brace such that

$$(\mathcal{B},+) = (G,+) \oplus \langle x \rangle \qquad (\mathcal{B},\circ) = (G,+) \rtimes_\lambda \langle x \rangle$$

with λ as the *lambda* function. Then

$$\mathcal{B} = \langle x, 1_0 \rangle \qquad \mathrm{Soc}(\mathcal{B}) = G.$$

Theorem

Let $(B, +, \circ)$ be a locally centrally nilpotent skew brace then the following are equivalent:

- B is i-noetherian;
- B is s-noetherian;
- B is a finitely generated skew brace.

A. Ballester-Bolinches, R. Esteban-Romero, M. Ferrara, V. Pérez-Calabuig, and M. Trombetti, "Central Nilpotency of Left Skew Braces and Solutions of the Yang–Baxter Equation", *Pacific J.Math.* 335 (2025), No. 1, 1–32.

M. Di Matteo, R. Esteban Romero, M. Ferrara and V. Pérez-Calabuig, "Chain Conditions on Skew Braces and Solutions of the Yang–Baxter Equation", arXiv:2607.28177v2

Minimal Conditions

McLain Theorem, 1956

A locally nilpotent group satisfies **min** if and only if it satisfies **min-n**.

DH. McLain, "On Locally Nilpotent Groups", *Math. Proc. Camb. Ph. Soc.* 52 (1956), No. 1 5–11

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Theorem

A locally centrally nilpotent **two-sided** skew brace is i-artinian if and only if it is s-artinian. In this case, the skew brace is also hypercentral and soluble.

Solutions of the Yang–Baxter Equation

Chain Conditions on Solutions

Definition

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Let max-con (respectively min-con) be the class of solutions satisfying the maximal (respectively minimal) condition on congruences.

M. Di Matteo, R. Esteban Romero, M. Ferrara and V. Pérez-Calabuig, “Chain Conditions on Skew Braces and Solutions of the Yang–Baxter Equation”, arXiv:2606.28177

Local Multipermutability

Definition

A solution (X, r) is **locally (hyper)multipermutational** if any finitely generated subsolution is (hyper)multipermutational.

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Let (X, r) be a solution, then the following are equivalent

- X is locally (hyper)multipermutational;
- $G(X, r)$ is locally (hyper)multipermutational.

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A solution (X, r) is **locally (hyper)multipermutational** if any finitely generated subsolution is (hyper)multipermutational.

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Let (X, r) be a solution, then the following are equivalent

- X is locally (hyper)multipermutational;
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Question

If (X, r) is a locally multipermutational solution, is there a relationship between max-con and max-sub? What about min-sub and min-con?

Lyubashenko solutions

Definition

Consider a non empty set X and f, g two permutations in X such that $f \circ g = g \circ f$, and

$$r : X \times X \longrightarrow X \times X$$

$$r(x, y) = (f(y), g(x))$$

then (X, r) is called a ***Lyubashenko solution***.

The class of Lyubashenko solutions is exactly the class of multipermutational solutions of level 1.

Chain conditions in Lyubashenko solutions

Theorem

Let (X, r) be a Lyubashenko solution, then the following are equivalent:

1. (X, r) is finitely generated;
2. (X, r) satisfies max-sub;
3. (X, r) satisfies min-sub;
4. (X, r) satisfies max-con;
5. X has a finite number of orbits under the action of the group $\langle f, g \rangle$.

Chain conditions in Lyubashenko solutions

Theorem

Let (X, r) be a Lyubashenko solution, then the following are equivalent:

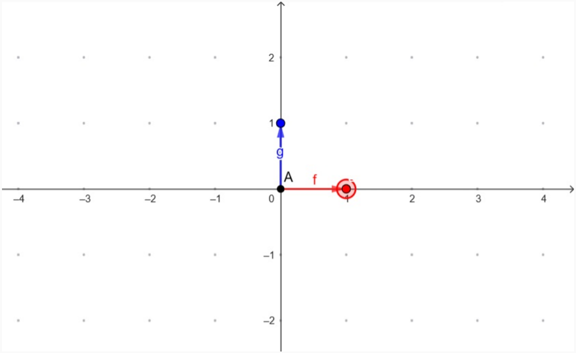
1. (X, r) is finitely generated;
2. (X, r) satisfies max-sub;
3. (X, r) satisfies min-sub;
4. (X, r) satisfies max-con;
5. X has a finite number of orbits under the action of the group $\langle f, g \rangle$.

Theorem

Let (X, r) be a Lyubashenko solution satisfying min-con, then it is finite.

Example

Let $X = \mathbb{Z} \times \mathbb{Z}$, $f((x_1, x_2)) = (x_1 + 1, x_2)$ and $g((x_1, x_2)) = (x_1, x_2 + 1)$. Thus if $r(\underline{x}, \underline{y}) = (f(\underline{y}), g(\underline{x}))$, then (X, r) is an example of a one-generated Lyubashenko solution which does not satisfy min-con.



Thank you for your attention!

Questions?