# Finite Group Algebras

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Braiding new ground:
Vertex algebras meet the Yang-Baxter Equation

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# Two definitions of a group ring

**Definition**

Given a group $G$ and a ring $R$, define the group ring $RG$ as the set of all formal finite linear combinations of elements of $G$, with coefficients in $R$. So $RG = \{\sum a_g g | a_g \in R, g \in G\}$.
Define addition in the obvious way: $\sum a_g g + \sum b_g g = \sum (a_g + b_g) g$.
Define multiplication as $(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{g,h \in G} (a_g b_h)(gh)$.

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Given a group $G$ and a ring $R$, define the group ring as the set of all functions from $G$ to $R$ (with finite support).
Define addition pointwise $(f + g) : x \mapsto f(x) + g(x)$
and define multiplication using convolution of functions:
$(fg) : x \mapsto \sum_{uv=x} f(u)g(v) = \sum_{u \in G} f(u)g(u^{-1}x)$

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# Famous problems in Group Rings

Let $K$ be a field and let $G$ be a group.

- **The unit problem:**
  What is the structure of the group of units of $KG$? Can we construct this group? What is its connection with $G$?

- Let $G$ be a torsion-free group. The **unit conjecture** asserts that KG has no units other than the trivial units $a_g g$ ($a_g \in K, g \in G$). The unit conjecture was disproved by Gardam (2021) in characteristic 2, with the group taken to be the Promislow group $P$.

- The **zero-divisor conjecture** asserts that the group ring KG of a torsion-free group $G$ over a field K has no zero divisors (open problem)

- **The isomorphism problem:**
  If the group rings $KG$ and $KH$ are isomorphic (as rings), do $G$ and $H$ have to be isomorphic (as groups)? No. D. García-Lucas, L. Margolis, Á. del Río (2022) and Hertweck (2001)

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**Theorem (C. 2008)**

*$\mathbb{F}_5 C_4$ is isomorphic to $\mathbb{F}_5 C_2 \times C_2$. These algebras of order 625 are the smallest group algebras with the property that $KG \cong KH$ with $G$ and $H$ not isomorphic.*

Suppose that $K$ and $G$ are finite. If the characteristic of $K$ does not divide the order of $G$ then $KG$ is semisimple and finding its decomposition as a direct sum of matrix rings over fields is fairly straightforward.

It is more difficult to find the group of units of non-commutative, non-semisimple group algebras $FG$ where $F$ has characteristic $p$ and $G$ is not a $p$-group. The first such result was given in 2008:

**Theorem (C. and Gildea 2008)**

$$\mathcal{U}(\mathbb{F}_{3^k} D_6) \cong (({C_3}^{3k} \rtimes {C_3}^{k}) \rtimes C_{3^k-1}) \times C_{3^k-1}.$$

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The next result directs our study of derivations of commutative group algebras to the nonsemisimple case.

**Theorem**

*If $K$ and $G$ are finite and $G$ is abelian with the order of $K$ dividing the orger of $G$ then every derivation of $KG$ is trivial*
*(i.e. finite commutative semisimple group algebras have only trivial derivations).*

We have characterised the finite commutative non-semisimple derivations have been characterised.

This technique is also useful for finding the derivations of non-commutative group algebras.

In the next Theorem we determine all derivations of finite commutative group algebras.

**Theorem (C. and Hughes 2019)**

*Let $K$ be a finite field of positive characteristic $p$. Let $G \simeq H \times X$ be a finite abelian group, where $H$ is a $p$-regular group and $X$ is a $p$-group with the following presentation*

$$X = \langle x_1, \dots, x_n \mid x_k^{p^{m_k}} = 1,\ [x_k, x_l] = 1,\ \text{for all } k, l \in \{1, 2, \dots n\} \rangle,$$

*where $n, m_k$ are natural numbers. For $i, j \in \{1, \dots, n\}$, let $f_i \colon \{x_1, \dots, x_n\} \to KG$ be defined by*

$$f_i(x_j) = \begin{cases} 1 & \text{if } i = j \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

*Then $f_i$ can be uniquely extended to a derivation of $KG$ denoted by $\partial_i$. Moreover $Der(KG)$ is a vector space over $K$ with basis $\{g\partial_i \mid g \in G, i = 1, \dots, n\}$.*

**Theorem (C. and Hughes 2019)**

*If $n$ is even, $Der(\mathbb{F}_{2^m}D_{2n})$ has dimension $2n+4$ and a basis*

$$\{d_{x',y'} \mid (x',y') \in \{(\lambda y, 0),\ (x\omega y, \omega) \mid \lambda \in B_e(xy),\ \omega \in B_e(y)\}\}.$$

*If $n$ is odd, $Der(\mathbb{F}_{2^m}D_{2n})$ has dimension $\frac{3n+1}{2}$ and a basis*

$\{d_{x',y'} \mid (x',y') \in$
$\{((x^i+x^{-i})y, 0), ((1+x)y, 1), (0, y), (x(x^i+x^{-i})y, x^i+x^{-i}), (0, (x^i+x^{-i})y$

# Associative rings of Derivations?

The next results were first motivated by the observation that derivations of an algebra of characteristic $p$ form a restricted Lie algebra [Jacobson 1937].

In particular $d \in Der(A)$ implies $d^p \in Der(A)$, but $Der(A)$ is not closed under the obvious multiplication (composition of derivations).

So $Der(A)$ is a vector space where (repeated) $p$th powers are allowed.

It is well known that the derivations of an associative algebra form a Lie algebra. It is rarely the case that this set of derivations forms an associative algebra. When can this happen?

# The rings of derivations

Since the composition of derivations is rarely a derivation, this motivates the following question: what associative structure do derivations have?

**Theorem (C. and Hughes 2024)**

*Let $K$ be a finite field of characteristic $p > 0$ and let $G$ be a finitely generated group. Denote the vector space of derivations equipped with a product defined as composition by $(Der(KG), \circ)$.*

*$(Der(KG), \circ)$ is the zero ring if and only if $G$ is abelian and $KG$ is semisimple.*

*$(Der(KG), \circ)$ is a non-zero ring if and only if $p = 2$ and $G \simeq A \times P$ where $A$ is a finite abelian group of odd order and $P$ is either a cyclic 2-group or an infinite cyclic group.*

Go raibh maith agaibh!

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