Computing multiplicity-free fusion rings and categories

Joost Slingerland

Joint work with Gert Vercleyen

Braiding new ground:
Vertex algebras meet the Yang-Baxter Equation
Maynooth, Sep 18, 2026

Advertizing Results and Tools

ANYONWIKI https://anyonwiki.org/

Open-access database of fusion rings and fusion categories for mathematical physics research – webpage interface

Team at Purdue now: Gert Vercleyen, Colleen Delaney, Ralph Razzouk

ANYONICA https://github.com/**gert-vercleyen**/Anyonica

Package for finding and exploring properties of fusion rings and categories – currently uses Mathematica (will be Julia/Oscar)

Vercleyen, G., JKS.
"On low rank fusion rings." Journal of Mathematical Physics 64.9 (2023).

All fusion rings up to rank 9 for several multiplicities.

Vercleyen, G., JKS.
On low-rank multiplicity-free complex fusion categories, CMP to be published
(Or Read Gert's Thesis now arxiv:2405.20075)

All multiplicity-free pivotal fusion categories up to rank 7
(F symbols, R symbols, and pivotal coefficients)

Some AnyonWiki Snapshots

Table of Fusion Rings with properties

AnyonWiki Fusion Rings Fusion Categories Documentation About Contact Cite Sign In

Search by name... | Advanced | Multiplicity-Free | Categorifiable | Rank | $\mathcal{D}^2_{\mathrm{FP}}$ | Default (Name) | Download

Showing 50 of 547 fusion rings (filtered from 25,332 total) — Page 1 of 11

Name	Rank	$\mathcal{D}^2_{\mathrm{FP}}$	FC	PFC	UFC	BFC	MFC	Actions
$\mathrm{FR}^{1,1,0}_1$: Trivial	1	1	✓	✓	✓	✓	✓	
$\mathrm{FR}^{2,1,0}_1 : \mathbb{Z}_2$ +3	2	2	✓	✓	✓	✓	✓	
$\mathrm{FR}^{2,1,0}_2$: Fib +3	2	3.618	✓	✓	✓	✓	✓	
$\mathrm{FR}^{3,1,2}_1 : \mathbb{Z}_3$ +1	3	3	✓	✓	✓	✓	✓	
$\mathrm{FR}^{3,1,0}_1$: Ising +3	3	4	✓	✓	✓	✓	✓	
$\mathrm{FR}^{3,1,0}_2 : \mathrm{Rep}(D_3)$ +3	3	6	✓	✓	✓	✓	✗	
$\mathrm{FR}^{3,1,0}_3 : \mathrm{PSU}(2)_5$	3	9.296	✓	✓	✓	✓	✓	
$\mathrm{FR}^{4,1,0}_1 : \mathbb{Z}_2 \otimes \mathbb{Z}_2$ +1	4	4	✓	✓	✓	✓	✓	
$\mathrm{FR}^{4,1,2}_1 : \mathbb{Z}_4$ +2	4	4	✓	✓	✓	✓	✓	
$\mathrm{FR}^{4,1,2}_2 : \mathrm{TY}(\mathbb{Z}_3)$ +2	4	6	✓	✓	✓	✗	✗	
$\mathrm{FR}^{4,1,0}_2 : \mathrm{SU}(2)_3$ +1	4	7.236	✓	✓	✓	✓	✓	
$\mathrm{FR}^{4,1,2}_3 : \mathrm{Fib}(\mathbb{Z}_3)$ +1	4	8.303	✗	✗	✗	✗	✗	
$\mathrm{FR}^{4,1,0}_3 : \mathrm{Rep}(D_5)$ +1	4	10	✓	✓	✓	✓	✗	

Some AnyonWiki Snapshots

Other side of theTable of Fusion Rings...

AnyonWiki Fusion Rings Fusion Categories Documentation About Contact Cite Sign In

Name	Rank	$\mathcal{D}^2_{\mathrm{FP}}$	FC	PFC	UFC	BFC	MFC	Actions
$\mathrm{FR}^{8,1,6}_{10} : [I \trianglelefteq \mathbb{Z}_4]^{\mathrm{Id}}_{\mathbf{3}\vert 1}$	8	75.777	–	–	–	–	–	
$\mathrm{FR}^{7,1,0}_{1} : \mathrm{Adj}(\mathrm{SO}(16)_2)$	7	16	✓	✓	✓	✓	✗	
$\mathrm{FR}^{7,1,0}_{2}$	7	21.123	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{3}$	7	27.154	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{4}$	7	28.944	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{5}$	7	29.46	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{6} : \mathrm{Adj}(\mathrm{SO}(11)_2)$	7	22	✓	✓	✓	✓	✗	
$\mathrm{FR}^{7,1,0}_{7} : \mathrm{SU}(2)_6$	7	27.314	✓	✓	✓	✓	✓	
$\mathrm{FR}^{7,1,0}_{8}$	7	28	✓	✓	✓	✓	✓	
$\mathrm{FR}^{7,1,0}_{9}$	7	34.385	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{10}$	7	43.314	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{11}$	7	36.971	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{12}$	7	42	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{13}$	7	52.93	✗	✗	✗	✗	✗	
$\mathrm{FR}^{7,1,0}_{14} : \mathrm{PSU}(2)_{12}$	7	70.685	✓	✓	✓	✓	✗	

Some AnyonWiki Snapshots

Inspect one of the rings... find the categories... (there is also a list of cats)

$\mathrm{FR}_2^{3,1,0} : \mathrm{Rep}(D_3)$

Download JSON

$\mathrm{Rep}(D_3)$ | $\mathrm{PSU}(2)_4$ | $\mathrm{Rep}(S_3)$ | $[\mathbb{Z}_2 \trianglelefteq \mathbb{Z}_2]^{\mathrm{Id}}_{1|1}$

Basic Properties

Rank	$\mathcal{D}^2_{\mathrm{FP}}$ (Numeric)	$\mathcal{D}^2_{\mathrm{FP}}$ (Algebraic)	AnyonWiki Code
3	6	6	[3, 1, 0, 2]

Commutative: Yes	Group ring: No	Nilpotent: No	Simple: No
Integral: Yes	Weakly integral: Yes	Non-trivially graded: No	

Categorifications

The following (pivotal) (braided) fusion categories categorify this fusion ring. They are indexed by three subscripts f, r, p: the f index distinguishes inequivalent associators (F-symbols); given f, the r index distinguishes inequivalent braidings (R-symbols); and given f, r, the p index distinguishes inequivalent pivotal structures (P-symbols).

Category	Spherical	Unitary	Braided	Ribbon	Modular
$\mathrm{FC}^{3,1,0}_{2,1,2,1} : [\mathrm{Rep}(D_3)]_{1,2,1}$	✓	✓	✓	✓	–
$\mathrm{FC}^{3,1,0}_{2,1,1,1} : [\mathrm{Rep}(D_3)]_{1,1,1}$	✓	✓	✓	✓	–
$\mathrm{FC}^{3,1,0}_{2,2,0,1} : [\mathrm{Rep}(D_3)]_{2,0,1}$	✓	✓	–	–	–
$\mathrm{FC}^{3,1,0}_{2,1,3,1} : [\mathrm{Rep}(D_3)]_{1,3,1}$	✓	✓	✓	✓	–
$\mathrm{FC}^{3,1,0}_{2,3,0,1} : [\mathrm{Rep}(D_3)]_{3,0,1}$	✓	✓	–	–	–

Some AnyonWiki Snapshots

Inspect one of the categories... find the full structure...

$$\mathrm{FC}^{3,1,0}_{2,1,2,1} : [\mathrm{Rep}(D_3)]_{1,2,1}$$

F-symbols

Associativity constraints (6j symbols)

Filter by label, e.g. 2_2_2

List Matrix

$[F^{3,3,3}_3]$

	1	2	3
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$
3	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	0

$[F^{1,1,1}_1]$

	1
1	1

R-symbols

Braiding constraints

3_3

List Matrix

Label	Exact
$[R^{1,3}_3]$	1
$[R^{2,3}_3]$	-1
$[R^{3,3}_1]$	$-\frac{1}{2} - \frac{1}{2}\sqrt{3}i$
$[R^{3,3}_2]$	$\frac{1}{2} + \frac{1}{2}\sqrt{3}i$
$[R^{3,3}_3]$	$-\frac{1}{2} + \frac{1}{2}\sqrt{3}i$

Notice that this category has
Nontrivial braiding (not permutation)

ANYONICA

Mathematica package to work with fusion systems

Contains

- Over 200 functions to create, query, manipulate, and combine fusion systems
- A dataset of 25000+ fusion rings and all multiplicity-free (pivotal) fusion systems up to rank 7
- Tools to find new fusion systems: **SolvePentagonEquations** and more
- More generally: tools to reduce and solve sparse systems with a huge number of polynomial equations: **FindZeroValues, ReduceByLinearity,** and more

Gert Vercleyen

Boiling Rings and Categories down to Numbers

Anyon Model

Unitary Modular Fusion System*

Fusion Ring
-
How to combine particles
-
Finite 3D table of integers
-

F-matrices
-
How to rearrange combinations
-
Finite set of finite invertible matrices
-

Pivotal Structure
-
How to deal with antiparticles
-
Finite vector of complex numbers
-

Braided Structure
-
How to deal with swapping particles
-
Finite set of finite invertible matrices
-

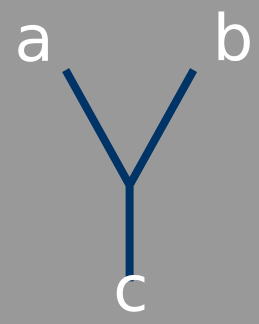
Depending on satisfied conditions, certain adjectives are used:

Unitary, Braided, Spherical, Ribbon, Modular

*arXiv:1305.2229

Searching Fusion Rings (step 1...)

Fusion Rules



$$a \times b = \sum_c N_c^{ab} c$$

Here N_c^{ab} is the dimension of the "space of states for two particles with topological charges a,b" such that the overall charge is c (*Multiplicity m* means all $N_c^{ab} \leqslant m$.)

There is a unit/vacuum/chargeless sector 1, with $N_b^{a1} = N_b^{1a} = \delta_{ab}$

For each charge, a there is a conjugate charge $\hat{a}$ with $N_1^{a\hat{a}} = N_1^{\hat{a}a} = 1$

Also, we require $N_c^{ab} = N_b^{\hat{a}c} = N_{\hat{a}}^{b\hat{c}} = N_{\hat{c}}^{\hat{b}\hat{a}} = N_{\hat{b}}^{a\hat{c}} = N_a^{\hat{b}c}$

and most importantly, Associativity (order of fusion doesn't matter)

$$\Sigma_e N_e^{ab} N_d^{ec} = \Sigma_f N_f^{bc} N_d^{af}$$

Now can do a brute force search for fusion rules with r types of charges (rank r), guessing N-values in {0,1,...,m}

But: Search space grows like $(m+1)^{r^3}$ (argl! Could only do up to r=5 even with m=1)

Smarter Search for Fusion Rings

First, break the permutation symmetry of the problem:

- Order the particles so selfdual ones come before non-selfdual ones and the nsd appear in pairs.
  (i.e. Fix the duality stucture – also good for implementing the pivotal identities ($N^{ab}{}_c = N^{\hat{a}c}{}_b$ etc.)

- Order the selfdual and non-selfdual ptcls by the sum of the entries in their fusion matrices,

So add these inequalities to the requirements on N:
$$\sum_{\substack{b=1\\c=1}}^{r} [N_i]^c_b \leq \sum_{\substack{b=1\\c=1}}^{r} [N_{i+1}]^c_b$$

Then run a backtracking algorithm filling in the constraints

(*)Find a constraint on N with the fewest possible number of variables.
  Call the set of variables V_1.
  Find all constraints that depend only on the variables in V_1
  Call that set of equations C_1
(**) Assume all variables in V_1 known and repeat (*) on the remaining variables and constraints to obtain V_2 and C_2.
Iterate to find more V_i and C_i until all variable are exhausted.
(***) Fill in variables like this: if you get all the way down you have a Fusion Ring

```
for V[1,1] in 0:m, V[1,2] in 0:m,...,V[1,-1] in 0:m
   if( all constr in C[1] are verified )
   for V[2,1] in 0:m, V[2,2] in 0:m,...,V[2,-1] in 0:m
      if( all constr in C[2] are verified )
      ...
         for V[k,1] in 0:m, V[k,2] in 0:m,...,V[k,-1] in 0:m
            if( constraints in C[k] are verified )
            saveSol( { V[1,1], V[1,2], ..., V[k,-1] } )
```

Note:
- No need to fill in entire N-matrices
- This will also work (better) with extra constraints

Fusion rings found (so far)

Total number of fusion rings found > 25000

	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**4**	**10**	**16**	**39**	**43**	**96**	**142**	**194**
2	**0**	**1**	**3**	**17**	**37**	**154**	**319**	*874*		
3	**0**	**1**	**4**	**24**	**82**	**384**	*562*			
4	**0**	**1**	**6**	**45**	**134**	**872**	*1236*			
5	**0**	**1**	**5**	**55**	**209**	*533*				
6	**0**	**1**	**9**	**81**	**336**	*872*				
7	**0**	**1**	**6**	**92**	**477**	*976*				
8	**0**	**1**	**10**	**137**	**733**	*1672*				
9	**0**	**1**	**12**	**151**	**863**					
10	**0**	**1**	**9**	**186**	**1082**					
11	**0**	**1**	**10**	**238**	**1383**					
12	**0**	**1**	**20**	**291**	**1948**					
13	**0**	**1**	**9**	**246**	*1300*					
14	**0**	**1**	**13**	**340**	*1323*					
15	**0**	**1**	**16**	**349**	*1550*					
16	**0**	**1**	**25**	**525**	*1925*					

Number of fusion rings found by rank (column) and highest fusion multiplicity (row)

Note:
the *Italic* numbers are not -total- numbers of rings
(did not complete full search for these cases)
The **bold** numbers are totals

Fusion Ring numbers by rank, multiplicity and number of non-self-dual particles

Numbers tend to increase with rank, except in some special cases (?)

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 0 && Mult[r] == q]], {p, 1, 9}, {q, 1, 16}]
```

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	4	6	5	9	6	10	12	9	10	20	9	13	16	25
6	15	22	42	53	79	89	134	149	183	236	287	244	338	346	522
10	28	67	115	192	310	448	696	1432	1751	2241	2979	1242	1275	1498	1843
20	102	286	703	313	532	611	1085	0	0	0	0	0	0	0	0
18	194	208	474	0	0	0	0	0	0	0	0	0	0	0	0
38	428	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

All selfdual

Rank 2: ψ x ψ = 1+ n ψ

(errors in rank 5, multiplicities>8 now corrected on wiki)

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 2 && Mult[r] == q]], {p, 3, 9}, {q, 1, 16}]
```

latrixForm=

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	2	3	2	2	3	3	2	3	2	4	2	2	3	3
5	8	15	19	17	26	29	37	31	43	42	70	58	48	52	82
11	39	82	152	202	312	342	557	0	0	0	0	0	0	0	0
17	91	277	642	0	0	0	0	0	0	0	0	0	0	0	0
30	276	0	0	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Rank 3: only Z_3

Rank 4: single equation in 3 variables:

$N(2,2,3)^2 + N(2,3,3)^2 = 1 + N(2,3,3) + N(2,2,3)\, N(3,3,3)$

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 4 && Mult[r] == q]], {p, 5, 9}, {q, 1, 16}]
```

latrixForm=

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	13	16	17	18	28	23	30	0	0	0	0	0	0	0	0
7	32	76	120	0	0	0	0	0	0	0	0	0	0	0	0
18	136	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Rank 5, mult 2:

Representation ring of smallest odd order non-Abelian group.

Dimensions: {1,1,1,3,3} (order 21)

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 6 && Mult[r] == q]], {p, 7, 9}, {q, 1, 16}]
```

latrixForm=

1	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
10	34	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Rank 7, mult 2:

Both have dimensions: {1,1,1,3,3,3,3}.

But there is only one non-Abelian group of order 39.... mult 3: dims {1,1,1,1,1,5,5}

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 8 && Mult[r] == q]], {p, 9, 9}, {q, 1, 16}]
```

latrixForm=

2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Z_9 and $Z_3 \times Z_3$

The “Million Dollar” Question:

Which of our rings correspond to fusion categories?

(most do not, but which do? Try to solve pentagon equations.)

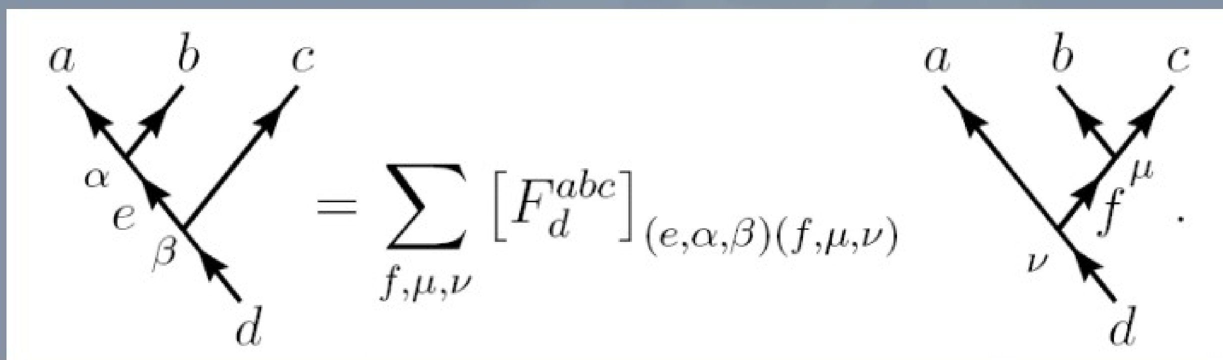
Or even complete Anyon Models or TQFTs with well-defined braiding?

(These questions are usually easier to answer, in the negative at least)

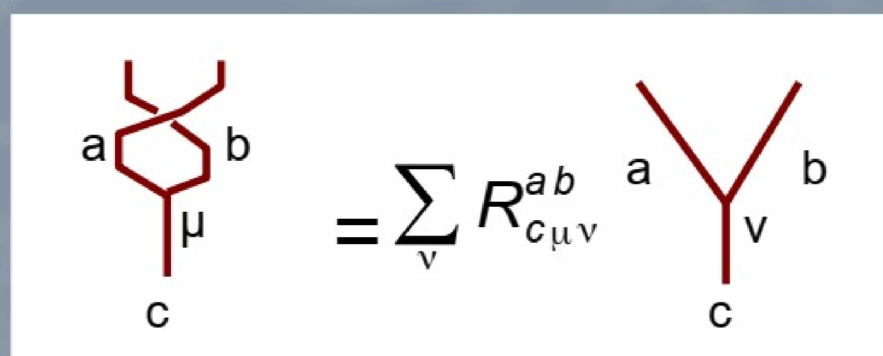
Mathematics of Anyon Models

An anyon model is mathematically a *Unitary Braided Tensor Category.*
(definition not in this talk)
The information in an anyon model can be summarized in F-symbols and R-symbols

F-matrix / F-symbols
Coefficients of change of basis (*recoupling*) in 3-anyon splitting spaces



R-symbols: Braiding acting on splitting spaces:



With the F- and R-symbols, we can calculate the amplitude for any fusion/braiding process

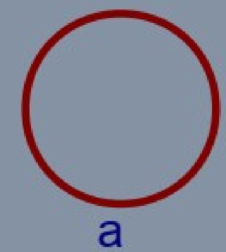
Reduction Strategy

Strategy: Remove loops (surplus vertices) to reach a tree.
Then move branches around to get the chosen standard tree

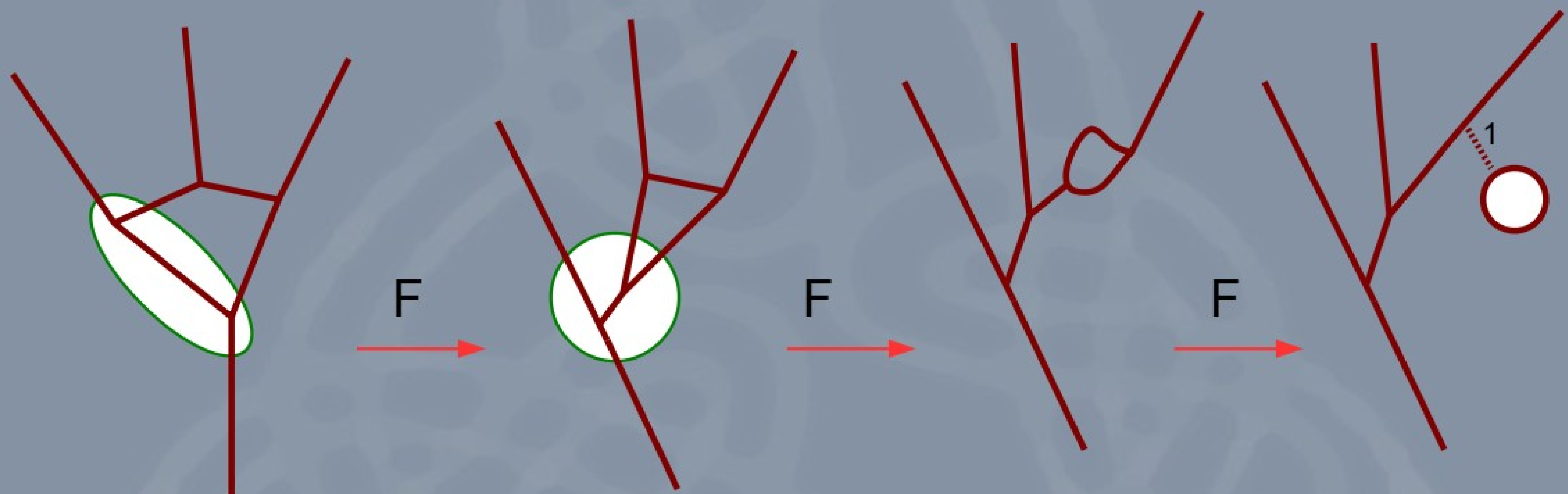
Note: tadpoles vanish

$= 0$

but



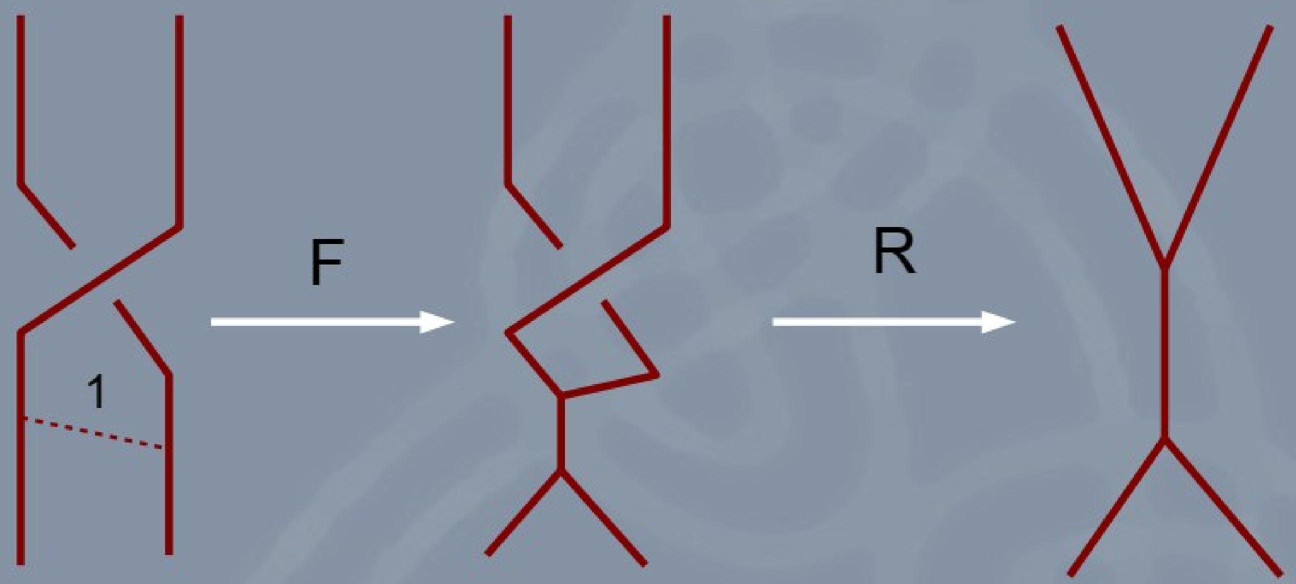
$= d_a$



Two vertices removed at the end (only the term with a disconnected loop survives);
We can build an algorithm for complete reduction out of this.

Removing Crossings

Need F-Symbols and R-symbols for this

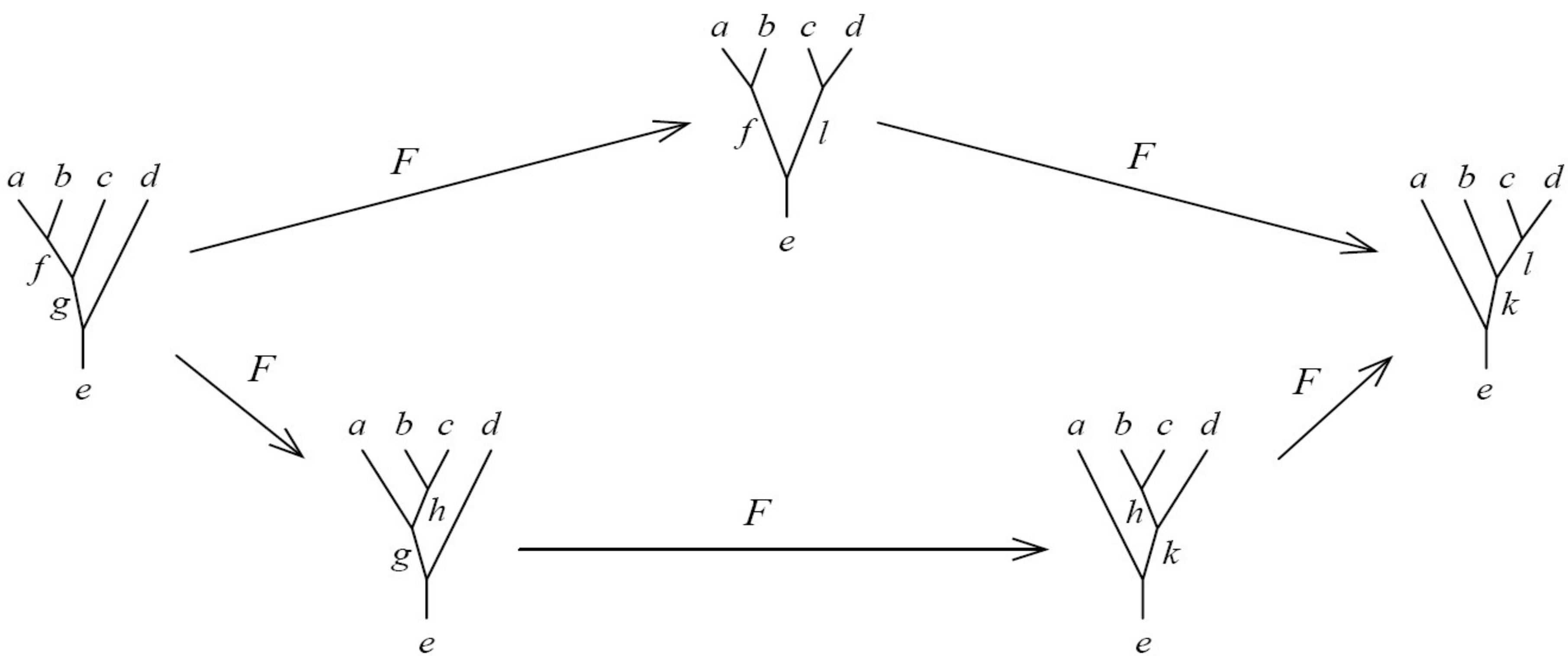


The two extra crossings can be removed by F-moves later on.

The suggested algorithm is very slow in general
(the number of terms generated is likely exponential in vertices/crossings)

But that's why we can have/need the quantum computer :)

The Pentagon Equations – Consistency condition for combining local F-moves

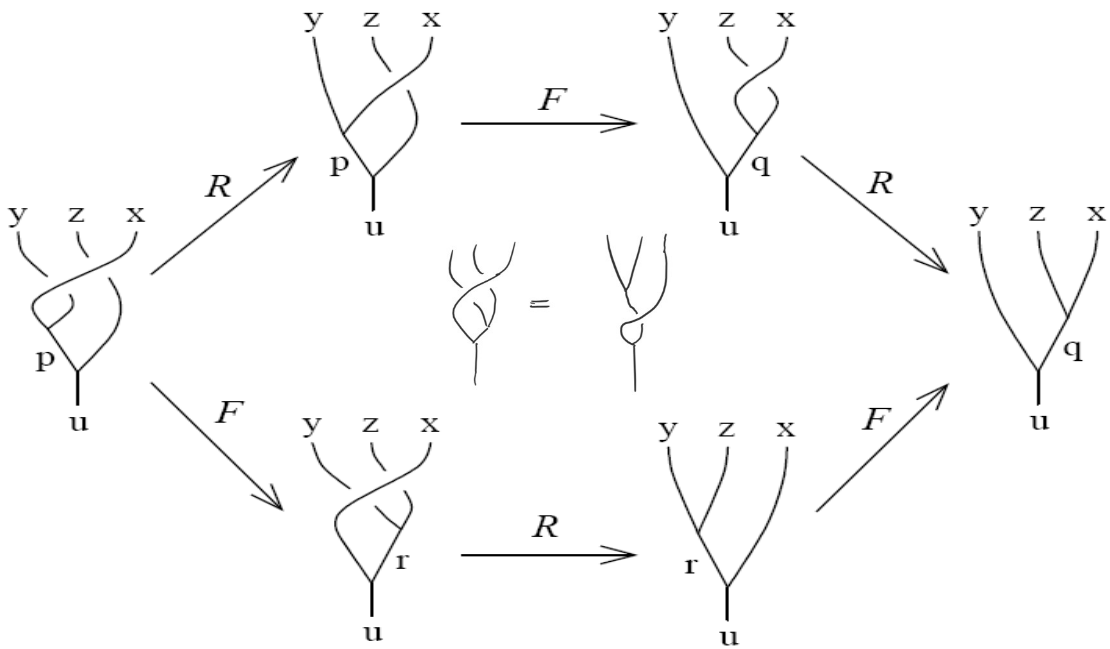


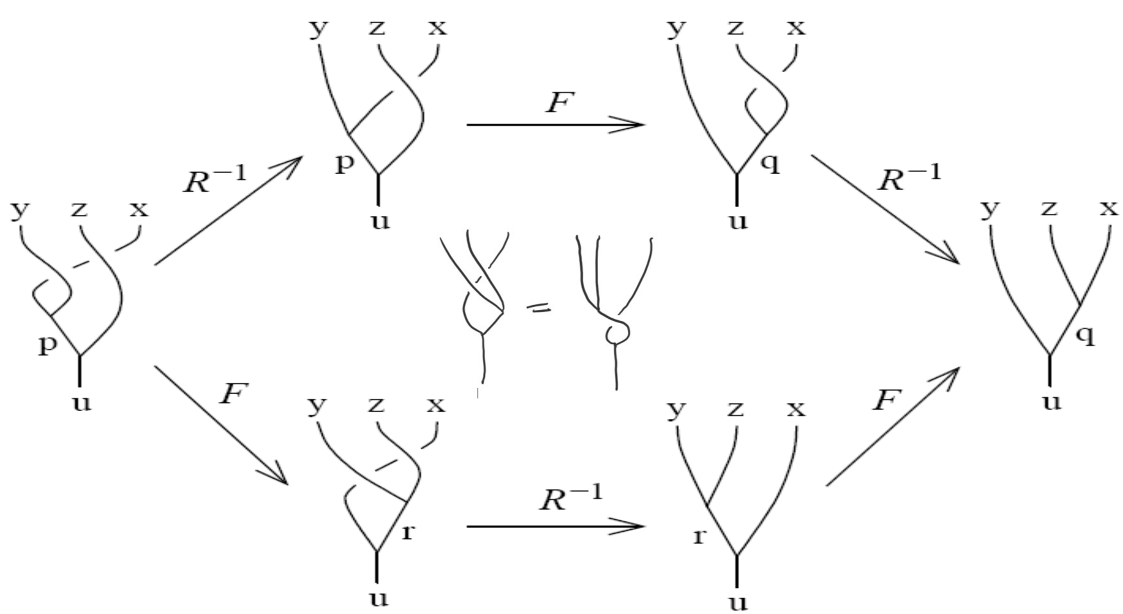
$$\left[F^{fcd}_{e}\right]_{gl}\left[F^{abl}_{e}\right]_{fk} = \sum_{h}\left[F^{abc}_{g}\right]_{fh}\left[F^{ahd}_{e}\right]_{gk}\left[F^{bcd}_{k}\right]_{hl}$$

Can write down Pentagons knowing only the fusion rules, then try to solve for F-symbols.
Similarly for the “Hexagon equations”, which also involve R-symbols

We mostly assume fusion multiplicities are < 2 ("no multiplicities"), so no vertex labels.
For this case we now have a solver that is in principle algorithmic, though with terrible scaling.
(We are still hoping but not sure how to build a solver with multiplicities)

Consistency of Braiding and Fusion: the Hexagon Equations





Properties of the pentagon (and hexagon) equations

- Third order polynomial equations in many variables
- Many more equations than variables: solutions do not always exist
- "Gauge" freedom (basis choices for vertex spaces)
  gives parameter families of equivalent solutions

Ocneanu rigidity:
only discrete set of solutions can exist modulo gauge freedom

Once gauge freedom is fixed the equations can (in principle)
be solved algorithmically, using Groebner bases (uses rigidity)

- This scales very badly with the number of variables
- However, the number of variables can be drastically reduced by using that
  many equations are linear in at least one variable

Gauge fixing with good properties, solver of the pentagon/hexagon etc. for Multiplicity Free theories
was implemented in a Mathematica program already starting 2005/6 (on laptop).

However, problem with zero F-symbols (next slide) - now resolved :)

Fixing the Gauge

Different choice of basis vectors in splitting spaces gauge transforms F and R

$$\left[F^{abc}_d\right]'_{ef} = \frac{u^{af}_d u^{bc}_f}{u^{ab}_e u^{ec}_d}\left[F^{abc}_d\right]_{ef} \qquad \left[R^{ab}_c\right]' = \frac{u^{ba}_c}{u^{ab}_c} R^{ab}_c .$$

Basic Idea

Can set $\left[F^{abc}_d\right]'_{ef} = c \neq 0$ by choice of one u, e.g. $u^{ab}_e = \dfrac{u^{af}_d u^{bc}_f \left[F^{abc}_d\right]}{u^{ec}_d c}$

Now eliminate u from the equations for the F', choose a different F-symbol and repeat. Iterate until all u's are eliminated.

Problem: What if $\left[F^{abc}_d\right]_{ef} = 0$? (...next slide!)

Note: all this is -much- more complicated with multiplicities

Finding zero F-symbols

Old trick: Can actually find absolute values of the F-symbols in many small examples, using unitarity

$$\sum_e \left| \left[F_d^{abc}\right]_{ef} \right|^2 = \sum_f \left| \left[F_d^{abc}\right]_{ef} \right|^2 = 1$$

and sum-free pentagon equations (sum over h has one term)

$$\left| \left[F_e^{fcd}\right]_{gl} \right|^2 \left| \left[F_e^{abl}\right]_{fk} \right|^2 = \left| \left[F_g^{abc}\right]_{fh} \right|^2 \left| \left[F_e^{ahd}\right]_{gk} \right|^2 \left| \left[F_k^{bcd}\right]_{hl} \right|^2 .$$

Can then set F'=|F| during gauge fixing procedure (i.e. c=|F|).

Then the resulting gauge has good properties

- If F-matrices can be unitary, they will be
- If F-matrices can be real, they will be

This trick does not always work (can't always fix |F|s) + Want to go beyond Unitary case.

ANYONICA finds a finite number of sets of F-symbols which might consistently be zero (can say more). Then tries solving with each possible set equal to zero. This no longer uses unitarity.

Can then attack Pentagon/Hexagon algortihmically – Sum free equations first!
Similar elimination of variables (Fs, Rs) as for gauge fixing followed by standard solution techniquess

Can still find a unitary gauge when it exists.

Excluding existence of Modular/Braided/just tensor categories

Without directly attacking the pentagon etc.

Modular Tensor Categories have a modular S matrix which
- simultaneously diagonalizes the fusion matrices
- is symmetric and unitary
- satisfies $S^2=C$ (charge conjugation, so $S^4=1$)

Also, there is a diagonal T-matrix with $(ST)^3=S^2$ (we ignore this here)

It's easy to check if such a matrix exists for any set of fusion rules.
- Find the fusion characters
  (diagonalize a random linear combination of the fusion matrices)
- Reorder and renormalize them in all possible orders
- See if one of the orders satisfies the requirements on S above

Also, modular data (S,T) known up to rank 12 (Ng, Rowell, Wen, arXiv:2308.09670)

Non-Modular, but still Braided Tensor Cats
- must have a symmetric subcategory ("transparent objects", Mueger!)
- such symmetric cats are representation cats of finite groups (Dopplicher-Roberts)

We can easily identify absence of subrings which are representation rings of groups
(Then not braided, if not modular)

Obstructions to just tensor cat-structure also in work of Liu/Palcoux/Wu

Numbers of Rings with S-matrices, by multiplicity, rank and number of non-self-duals

Multiplicity free

1	2	2	4	2	6	3	9	6
		1	1	1	1	2	2	2
				1	3	0	2	2
						1	1	2
								2

M=2

0	1	1	4	2	9	5	13	0
		0	0	1	0	1	0	0
				0	1	2	2	0
						0	0	0
								0

M=3

0	1	0	2	1	2	0	0	0
		0	0	0	0	0	0	0
				0	1	0	0	0
						0	0	0
								0

0	1	0	3	0	3	0	0	0
		0	0	1	0	0	0	0
				0	1	0	0	0
						0	0	0
								0

0	1	0	2	0	0	0	0	0
		0	0	0	1	0	0	0
				0	1	0	0	0
						0	0	0
								0

0	1	0	3	0	1	0	0	0
		0	0	0	1	0	0	0
				0	1	0	0	0
						0	0	0
								0

We have (only) 177 “candidate MTCs” total up to rank 9

There are cases where S exists, but there is no MTC (can check for existence of T-matrix)

- Some trivial (rank 2, most of the others)

- Some less trivial (interesting :))

118 non-Abelian Fusion Rings up to rank 9 (+53 at rank 10, mult free)

...For objects in 1D with non-commutative fusion!

```
Table[Length[Select[nonabrings, (Rank[#] == p && Mult[#] == m) &]], {m, 1, 8}, {p, 6, 9}]
```

MatrixForm=

$$\begin{pmatrix} 2 & 3 & 19 & 26 \\ 2 & 5 & 40 & 0 \\ 1 & 6 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

By mult/rank

2 0	3 0 0	13 3 3	11 12 3 0	M=1
2 0	5 0 0	30 5 5	0 0 0 0	M=2
1 0	6 0 0	0 0 0	0 0 0 0	M=3
1 0	6 0 0	0 0 0	0 0 0 0	
1 0	0 0 0	0 0 0	0 0 0 0	
2 0	0 0 0	0 0 0	0 0 0 0	
2 0	0 0 0	0 0 0	0 0 0 0	
2 0	0 0 0	0 0 0	0 0 0 0	

By mult/rank split by number of non-self-dual particles (2,4,6,8)

Rank 6 non-Abelian Fusion Rings

Multiplicity free:

- We have the smallest non-Abelian group D_3 (or S_3)
- The Haagerup-Izumi theory $HI(Z_3)$, fusion rules:

$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$	$\psi[5]$	$\psi[6]$
$\psi[2]$	$\psi[1]+\psi[2]+\psi[3]+\psi[4]$	$\psi[2]+\psi[3]+\psi[4]+\psi[6]$	$\psi[2]+\psi[3]+\psi[4]+\psi[5]$	$\psi[4]$	$\psi[3]$
$\psi[3]$	$\psi[2]+\psi[3]+\psi[4]+\psi[5]$	$\psi[1]+\psi[2]+\psi[3]+\psi[4]$	$\psi[2]+\psi[3]+\psi[4]+\psi[6]$	$\psi[2]$	$\psi[4]$
$\psi[4]$	$\psi[2]+\psi[3]+\psi[4]+\psi[6]$	$\psi[2]+\psi[3]+\psi[4]+\psi[5]$	$\psi[1]+\psi[2]+\psi[3]+\psi[4]$	$\psi[3]$	$\psi[2]$
$\psi[5]$	$\psi[3]$	$\psi[4]$	$\psi[2]$	$\psi[6]$	$\psi[1]$
$\psi[6]$	$\psi[4]$	$\psi[2]$	$\psi[3]$	$\psi[1]$	$\psi[5]$

With multiplicity:

- simple generalizations of Haagerup, with n times (psi[3]+psi[4]+psi[5])

$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$	$\psi[5]$	$\psi[6]$
$\psi[2]$	$\psi[1]+5\psi[2]+5\psi[3]+5\psi[4]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[6]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[5]$	$\psi[4]$	$\psi[3]$
$\psi[3]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[5]$	$\psi[1]+5\psi[2]+5\psi[3]+5\psi[4]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[6]$	$\psi[2]$	$\psi[4]$
$\psi[4]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[6]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[5]$	$\psi[1]+5\psi[2]+5\psi[3]+5\psi[4]$	$\psi[3]$	$\psi[2]$
$\psi[5]$	$\psi[3]$	$\psi[4]$	$\psi[2]$	$\psi[6]$	$\psi[1]$
$\psi[6]$	$\psi[4]$	$\psi[2]$	$\psi[3]$	$\psi[1]$	$\psi[5]$

Four others without subgroups, one of them particularly interesting at multiplicity two... (next slide)

Note: All other non-Abelian rings we have found (at rank>6) have a group as a subring

The rank 6 Fusion Hecke algebra at multiplicity 2

$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$	$\psi[5]$	$\psi[6]$
$\psi[2]$	$\psi[1]+\psi[2]$	$\psi[6]$	$\psi[4]+\psi[5]$	$\psi[4]$	$\psi[3]+\psi[6]$
$\psi[3]$	$\psi[5]$	$\psi[1]+\psi[3]$	$\psi[4]+\psi[6]$	$\psi[2]+\psi[5]$	$\psi[4]$
$\psi[4]$	$\psi[4]+\psi[6]$	$\psi[4]+\psi[5]$	$\psi[1]+\psi[2]+\psi[3]+2\psi[4]+\psi[5]+\psi[6]$	$\psi[3]+\psi[4]+\psi[5]+\psi[6]$	$\psi[2]+\psi[4]+\psi[5]+\psi[6]$
$\psi[5]$	$\psi[3]+\psi[5]$	$\psi[4]$	$\psi[2]+\psi[4]+\psi[5]+\psi[6]$	$\psi[4]+\psi[6]$	$\psi[1]+\psi[3]+\psi[4]$
$\psi[6]$	$\psi[4]$	$\psi[2]+\psi[6]$	$\psi[3]+\psi[4]+\psi[5]+\psi[6]$	$\psi[1]+\psi[2]+\psi[4]$	$\psi[4]+\psi[5]$

The elements $\mathbf{2}$ and $\mathbf{3}$ generate Fibonacci subrings. Elements $\mathbf{5} = \mathbf{3} \times \mathbf{2}$ and $\mathbf{6} = \mathbf{2} \times \mathbf{3}$ can be regarded as couples of Fibonacci particles and $\mathbf{4} = \mathbf{2} \times \mathbf{3} \times \mathbf{2} = \mathbf{3} \times \mathbf{2} \times \mathbf{3}$ as a tripple of Fibonacci particles. Indeed, the above fusion ring is completely determined by the generators $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$ together with the relations $\mathbf{2}^2 = \mathbf{1} + \mathbf{2}, \mathbf{3}^2 = \mathbf{1} + \mathbf{3}$, and $\mathbf{2} \times \mathbf{3} \times \mathbf{2} = \mathbf{3} \times \mathbf{2} \times \mathbf{3}$. The structure of the ring thus corresponds to that of a Hecke algebra with generators $\mathbf{2}$ and $\mathbf{3}$.

Q: Does anyone already know if there is a corresponding fusion category?

The rank 7 non-Abelian Fusion Rings.

(20 rings, 3 multiplicity free. These are not categorifiable.)

FP Quantum dimensions of all 20 rings below

```
FullSimplify[ToRadicals[QuantumDimensions /@ nonabsevens]] // MatrixForm
```

MatrixForm=

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & \sqrt{6} & 1 & 1 \\
1 & 1 & 1 & 1 & 3 & 1 & 1 \\
1 & \text{Root}[1-\#1-4\#1^2+\#1^3\&,3] & \text{Root}[1-\#1-4\#1^2+\#1^3\&,3] & \text{Root}[1-\#1-4\#1^2+\#1^3\&,3] & \text{Root}[9-6\#1-3\#1^2+\#1^3\&,3] & 1 & 1 \\
1 & 1 & 1 & 1 & 1+\sqrt{7} & 1 & 1 \\
1 & 2 & 2 & 2 & 3 & 1 & 1 \\
1 & 3 & 3+\sqrt{13} & 3+\sqrt{13} & 3+\sqrt{13} & 1 & 1 \\
1 & 2+\sqrt{6} & 2+\sqrt{6} & 2+\sqrt{6} & 3+\sqrt{6} & 1 & 1 \\
1 & 3 & 4 & 4 & 4 & 1 & 1 \\
1 & 4+\sqrt{19} & 4+\sqrt{19} & 4+\sqrt{19} & 5+\sqrt{19} & 1 & 1 \\
1 & 1 & 1 & 1 & \frac{1}{2}\left(3+\sqrt{33}\right) & 1 & 1 \\
1 & 3+\sqrt{15} & 4+\sqrt{15} & 4+\sqrt{15} & 4+\sqrt{15} & 1 & 1 \\
1 & 3 & \frac{1}{2}\left(9+\sqrt{97}\right) & \frac{1}{2}\left(9+\sqrt{97}\right) & \frac{1}{2}\left(9+\sqrt{97}\right) & 1 & 1 \\
1 & \text{Root}[9+3\#1-6\#1^2+\#1^3\&,3] & \text{Root}[1-4\#1-\#1^2+\#1^3\&,3] & \text{Root}[1-4\#1-\#1^2+\#1^3\&,3] & \text{Root}[1-4\#1-\#1^2+\#1^3\&,3] & 1 & 1 \\
1 & \text{Root}[3+5\#1-6\#1^2+\#1^3\&,3] & \text{Root}[3+5\#1-6\#1^2+\#1^3\&,3] & \text{Root}[3+5\#1-6\#1^2+\#1^3\&,3] & \text{Root}[9-24\#1-5\#1^2+\#1^3\&,3] & 1 & 1 \\
1 & 2\left(3+\sqrt{10}\right) & 2\left(3+\sqrt{10}\right) & 2\left(3+\sqrt{10}\right) & 7+2\sqrt{10} & 1 & 1 \\
1 & 5+\sqrt{34} & 6+\sqrt{34} & 6+\sqrt{34} & 6+\sqrt{34} & 1 & 1 \\
1 & \text{Root}[2-4\#1-14\#1^2+\#1^3\&,3] & \text{Root}[2-4\#1-14\#1^2+\#1^3\&,3] & \text{Root}[2-4\#1-14\#1^2+\#1^3\&,3] & \text{Root}[27+3\#1-11\#1^2+\#1^3\&,3] & 1 & 1 \\
1 & 1 & 1 & 1 & 2+\sqrt{10} & 1 & 1 \\
1 & \text{Root}[1-\#1-13\#1^2+\#1^3\&,3] & \text{Root}[1-\#1-13\#1^2+\#1^3\&,3] & \text{Root}[1-\#1-13\#1^2+\#1^3\&,3] & \text{Root}[18-6\#1-6\#1^2+\#1^3\&,3] & 1 & 1 \\
1 & 3 & 2\left(3+\sqrt{10}\right) & 2\left(3+\sqrt{10}\right) & 2\left(3+\sqrt{10}\right) & 1 & 1
\end{pmatrix}
$$

$TY(D_3)$
$Fib(D_3)$
...
All have Z_3 inside

Mult free #3

“Pseudo-Modular” Tensor Categories? (Zestings!)

A section from my 2006/7 notes:

B.24 $\mathrm{SU}(2)_4$ like fusion rules without hexagon solutions

$$\begin{array}{|cccccc|}\hline \psi_0 & \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_1 & \psi_0 & \psi_2 & \psi_4 & \psi_3 \\ \psi_2 & \psi_2 & \psi_0+\psi_1+\psi_2 & \psi_3+\psi_4 & \psi_3+\psi_4 \\ \psi_3 & \psi_4 & \psi_3+\psi_4 & \psi_1+\psi_2 & \psi_0+\psi_2 \\ \psi_4 & \psi_3 & \psi_3+\psi_4 & \psi_0+\psi_2 & \psi_1+\psi_2 \\ \hline \end{array} \qquad (70)$$

These fusion rules are similar to those for $SU(2)_4$, but the particles ψ_3 and ψ_4 (with quantum dimension $\sqrt{3}$) are not self-dual, but dual to each other. There are 2 solutions to the pentagon, forming 1 mirror pair, tabulated below. Both solutions are unitary and both are invariant under the automorphism of the fusion rules that exchanges ψ_3 with ψ_4. There are no solutions to the hexagon equations.

unitary	$\mathcal{D} = 2\sqrt{3}$				
	ψ_0	ψ_1	ψ_2	ψ_3	ψ_4
d	1	1	2	$\sqrt{3}$	$\sqrt{3}$
κ	1	1	1	0	0

(71)

```
Chop[2 Sqrt[3] modulist512[[3, 2, 1]]] // MatrixForm
```

latrixForm=

“S-matrix”:

$$\begin{pmatrix} 1. & 1. & 2. & 1.73205 & 1.73205 \\ 1. & 1. & 2. & -1.73205 & -1.73205 \\ 2. & 2. & -2. & 0 & 0 \\ 1.73205 & -1.73205 & 0 & 0.-1.73205\,i & 0.+1.73205\,i \\ 1.73205 & -1.73205 & 0 & 0.+1.73205\,i & 0.-1.73205\,i \end{pmatrix}$$

A nice class of Fusion Rings – "Song extensions"

(many non-Abelian, at least some cats)

Let G be a finite group, H a normal subgroup of G.

We take simple objects labeled by the elements g of G

With g1 x g2 = g1 . g2

These act on a single orbit of further objects, labeled by elements of G/H
There are |G|/|H| of these, notated as gH
(g is some element of the coset gH).

We have a further element g_0H of G/H and an automorphism
A: G/H → G/H,satisfying

$A(g_0H)=g_0H$

$A^{-1}(gH) = (g_0H)^{-1} A(gH)\, g_0H$

We then define:

g1 x g2 = g1 . g2 (as before)

g1 x g2H= (g1.g2)H

g1H x g2 = g1H . A(g2 H)

g1H x g2H = L (g1H . A(g2H) . g_0H) sum_{h in H} h

Where L(gH) is an arbitrary element of the coset gH inside G.
(the choice of lift into G doesn't matter)

This captures some of the
Non-Abelian rank 8 theories
(with $G=D_3$, $H=Z_3$)

Computer Solutions of a non-Abelian fusion category based on D3, with two non-group objects

656 possibilities for the outer labels on the F-symbol (=number of 'F-matrices')

648 F-symbols with only one fusion channel

455 F-symbols with only one fusion channel and no vac on the three upper lines

Number of F-symbols not fixed after the gauging step: 477

```
fs[psi[7], psi[6], psi[7], psi[6], psi[5], psi[5]] → a[1]
fs[psi[7], psi[7], psi[6], psi[6], psi[2], psi[3]] → a[2]
fs[psi[7], psi[7], psi[7], psi[7], psi[2], psi[1]] → a[3]
```

$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$

From prev slide:

$G=D3$, $H=Z3$, $A=Id$, $g_0H = H$

Easily solved by computer

16 unitary solutions up to gauge

(some probably equivalent under Fusion automorphism)

Probably can have Full Analytic solution (?)
(Similar to solving Tambara-Yamagami)

Rank	self-dual	2 non-selfdual	4 non-self-dual
1	trivial		
2	$\mathbb{Z}_2 \cong \mathbf{SU(2)_1}$ $\mathrm{Fib} \cong \mathbf{PSU(2)_3}$		
3	$\mathrm{Ising} \cong \mathbf{SU(2)_2}$ $\mathbf{PSU(2)_5}$	$\mathbb{Z}_3 \cong \mathbf{SU(3)_1}$	
no MTC	$\mathbf{Rep(D_3)} \cong \mathbf{PSU(2)_4}$		
4	$\mathbb{Z}_2 \times \mathbb{Z}_2$ $\mathbf{SU(2)_3} \cong \mathrm{Fib} \times \mathbb{Z}_2$ $\mathrm{Fib} \times \mathrm{Fib}$ $\mathbf{PSU(2)_7}$	$\mathbb{Z}_4 \cong \mathbf{SU(4)_1}$	
no MTC	$\mathbf{Rep(D_5)} \cong \mathbf{SO(5)_2}/\mathbb{Z}_2$ $\mathbf{PSU(2)_6}$	$\mathbf{TY}(\mathbb{Z}_3)$ pseudo $\mathbf{PSU(2)_6}$	
no FC		1 ring	
5	$\mathbf{SU(2)_4}$ $\mathbf{PSU(2)_9}$		$\mathbb{Z}_5 \cong \mathbf{SU(5)_1}$
no MTC	$\mathbf{Rep(D_4)} \cong \mathbf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ $\mathbf{Rep(S_4)}$ $\mathbf{Rep(D_7)} \cong \mathbf{SO(7)_2}/\mathbb{Z}_2$ $\mathbf{PSU(2)_8}$	$\mathbf{TY}(\mathbb{Z}_4)$ $\mathbf{Fib}(\mathbb{Z}_4)$ pseudo $\mathbf{SU(2)_4}$ pseudo $\mathbf{Rep(S_4)}$	
no FC	4 rings	1 ring	
6	$\mathbf{SU(2)_2} \times \mathbb{Z}_2$ $\mathbf{SU(2)_2} \times \mathbf{Fib}$ $\mathbf{SU(2)_5} \cong \mathbf{PSU(2)_5} \times \mathbb{Z}_2$ $\mathbf{PSU(2)_5} \times \mathbf{Fib}$ $\mathbf{SO(5)_2}$ $\mathbf{PSU(2)_{11}}$		$\mathbb{Z}_6 \cong \mathbf{SU(6)_1}$ $\mathbf{SU(3)_2} \cong \mathbf{Fib} \times \mathbb{Z}_3$
no MTC	$\mathbb{Z}_2 \times \mathbf{Rep(D_3)}$ $\mathrm{Fib} \times \mathbf{Rep(D_3)}$ $\mathbf{Rep}(\mathbb{Z}_3 \rtimes \mathbf{D_3})$ $\mathbf{Rep(D_9)}$ $\mathbf{PSU(2)_{10}}$	$\mathbf{D_3}$ $[\mathbb{Z}_2 \trianglelefteq \mathbb{Z}_2 \otimes \mathbb{Z}_2]^{\mathrm{Id}}_{3\|0}$ $[\mathbb{Z}_2 \trianglelefteq \mathbb{Z}_4]^{\mathrm{Id}}_{1\|0}$ $\mathbf{Rep(Dic_{12})}$ pseudo $\mathbf{SO(5)_2}$ $\mathbf{HI}(\mathbb{Z}_3)$	$\mathbf{TY}(\mathbb{Z}_5)$ $\mathbf{MR_6}$
no FC	9 rings	5 rings	4 rings

Multiplicity Free Fusion Rings up to rank 6 - overview

(MTCs above the first line at each rank)

We know all cats for all of these rings (+ rank 7)

Much more + more detail
- in paper/thesis
- on AnyonWiki
- with ANYONICA

Outlook/Wishlist

- Generate category information where not known using pentagon/hexagon solver
  Generalize the solver to deal with multiplicities!

- Find “Natural” constructions of the various non-Abelian Fusion Rings
  e.g. as boundary theories

- Implement “standard” constructions in Mathematica package,
  e.g. Drinfeld Center/Double, Anyon Condensation...

- Implement diagram reduction, ... quantities for physics modeling

- Look at various other classes of objects
  higher cats?

- Referencing (ouch!),
- Interface with other software
  (GAP, Kac), databases (nLab, VOA catalogue)...

- Get community involved :) → Thanks for listening!