Ito's conjecture and Supersymmetry

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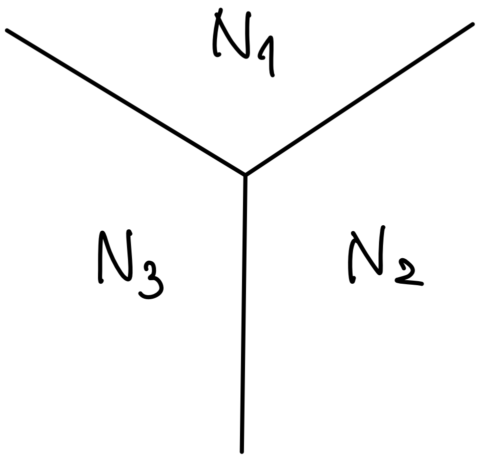
Braiding new ground: Vertex algebras meet the Yang-Baxter Equation
University of Maynooth

Based on Creutzig-Kovalchuk-Linshaw-S.-Suh, arXiv:2604.20750

Introduction



In 2017, Gaiotto and Rapčák introduced a family of vertex algebras $Y^{\varepsilon_1,\varepsilon_2,\varepsilon_3}_{N_1,N_2,N_3}$ called Y-algebras associated to interfaces of GL-twisted N=4 supersymmetric gauge theories.

They conjectured the triality of Y-algebras, namely,

$$Y^{\varepsilon_1,\varepsilon_2,\varepsilon_3}_{N_1,N_2,N_3} \cong Y^{\varepsilon_{\sigma(1)},\varepsilon_{\sigma(2)},\varepsilon_{\sigma(3)}}_{N_{\sigma(1)},N_{\sigma(2)},N_{\sigma(3)}} \qquad \forall \sigma \in S_3$$

In 2022, the triality was partially proven by Creutzig and Linshaw, when one of the indices N_1, N_2 and N_3 is given by 0.

This was a generalization of

Coset realization: $W^k(\mathfrak{sl}_n) \cong \mathrm{Com}(V^{\ell+1}(\mathfrak{sl}_n), V^\ell(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n))$, $\dfrac{1}{k+n} - \dfrac{1}{\ell+n} = 1$

Feigin-Frenkel duality: $W^k(\mathfrak{sl}_n) \cong W^\ell(\mathfrak{sl}_n)$, $(k+n)(\ell+n) = 1$

In general, the triality of Y-algebra is expected to be related with W-algebras associated with A-type Lie superalgebras.

The first step toward the triality would be Ito's conjecture

Ito's conjecture (1991) For $k \neq -1$,

$$W^k(\mathfrak{sl}_{n+1|n}) \cong \mathrm{Com}(V^{\ell+1}_{N=1}(\mathfrak{gl}_n), V^{\ell}_{N=1}(\mathfrak{sl}_{n+1}))$$

where $(k+1)(\ell+n+1) = 1$.

It was proven in my recent collaboration with Creutzig, Linshaw, Kovalchuk and Suh.

In this talk,

1. Understand the statement of Ito's conjecture
2. RHS; GKO construction and its supersymmetric analog
3. LHS; connection with $W^k(\mathfrak{sl}_{n+1|n})$

Understanding Ito's conjecture

$$W^k(\mathfrak{sl}_{n+1|n}) \cong \mathrm{Com}(V^{\ell+1}_{N=1}(\mathfrak{gl}_n), V^{\ell}_{N=1}(\mathfrak{sl}_{n+1}))$$

$$\mathfrak{sl}_{n+1|n} = \left\{ X = \begin{array}{cc} \overbrace{}^{\substack{\text{even} \\ n+1}} & \overbrace{}^{\substack{\text{odd} \\ n}} \end{array} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{array}{l} \}\, n+1 \\ \}\, n \end{array} \;\middle|\; \mathrm{str} X = \mathrm{tr} A - \mathrm{tr} D = 0 \right\}$$

$$\Psi$$

$$F_{prin} = \left[\begin{array}{cccc|cccc} 0 & & & & & & & \\ 1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ \hline & & & & 0 & & & \\ & & & & 1 & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{array}\right] \quad \text{even principal nilpotent}$$

$W^k(\mathfrak{sl}_{n+1|n})$ is defined by the quantum Hamiltonian reduction of $V^k(\mathfrak{sl}_{n+1|n})$ along F_{prin}, conjecturally a certain truncation of affine Yangian.

As a vertex algebra, $W^k(\mathfrak{sl}_{n+1|n})$ has free generators

$$\{\, w(a) \mid a \in \operatorname{Ker}(\operatorname{ad} F_{prin}) \,\}$$

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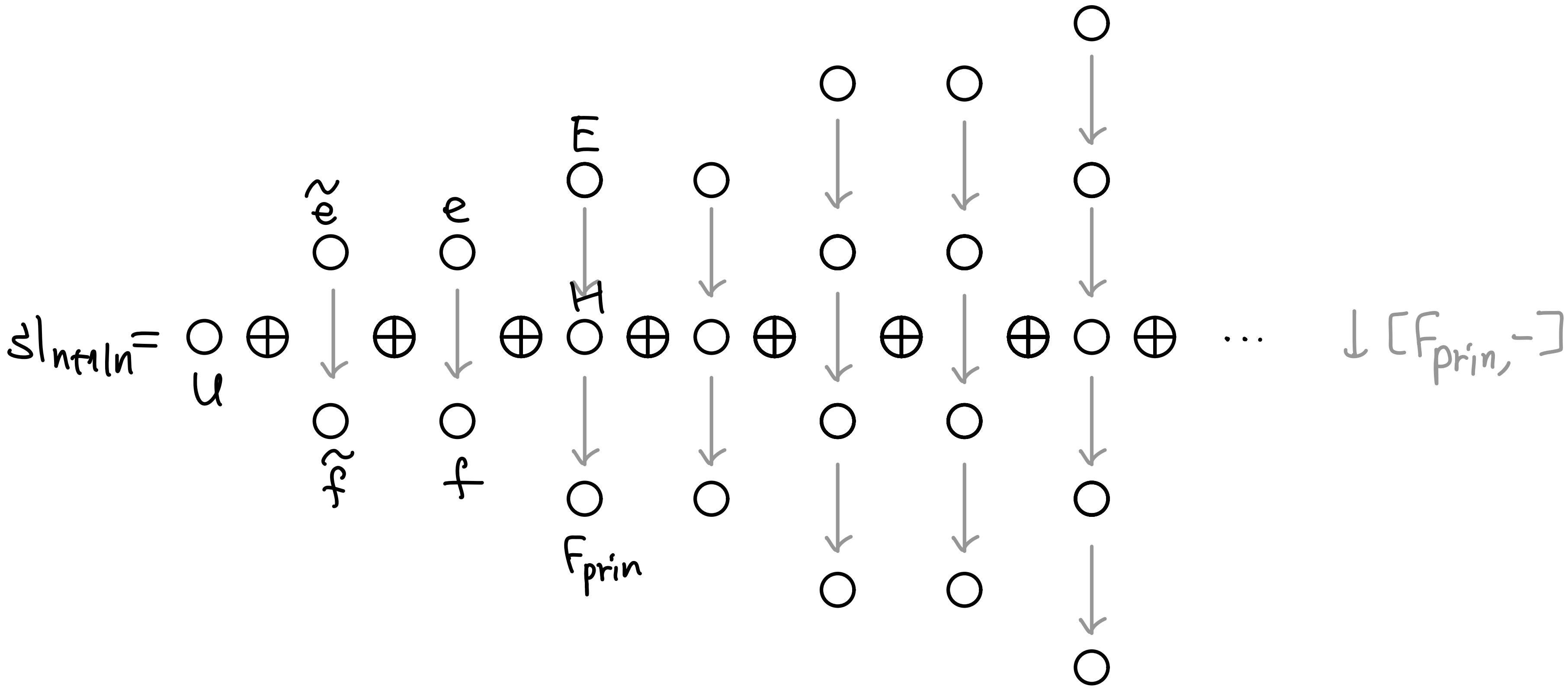
$$\{ w(a) \mid a \in \mathrm{Ker}(\mathrm{ad}\, F_{prin}) \}$$

Extend F_{prin} to an $\mathfrak{sl}_2$-triple $(E, H, F_{prin}) \subset \mathfrak{sl}_{n+1|n}$ and decompose $\mathfrak{sl}_{n+1|n}$ into $\mathfrak{sl}_2$-irreducible modules



Then $W^k(\mathfrak{sl}_{n+1|n})$ a vertex algebra freely generated by

$$w(u), \; w(\tilde{f}), \; w(f), \; w(F_{prin}), \; \cdots$$

$$W^k(\mathfrak{sl}_{n+1|n}) \cong \mathrm{Com}\left(V^{\ell+1}_{N=1}(\mathfrak{gl}_n), V^{\ell}_{N=1}(\mathfrak{sl}_{n+1})\right)$$

In a vertex algebra V, write each field by

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \quad \in (\mathrm{End}\, V)[[z^{\pm 1}]]$$

An odd endomorphism D on V is called a <u>supersymmetry</u> if

(i) $D^2 = \partial$ for the translation operator ∂ on V,

(ii) D is an odd derivation on V, that is,

$$[D, a_{(n)}] = (Da)_{(n)} \quad \forall a \in V,\ \forall n \in \mathbb{Z}.$$

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Definition (Heluani-Kac (2007))

A vertex algebra with n supercommuting supersymmetries is called an $N=n$ supersymmetric (SUSY) vertex algebra.

We only consider $N=1$ or $N=2$ SUSY vertex algebras

Recall that on a vertex algebra, we have

$$[\partial, a_{(n)}] = -n\, a_{(n-1)} \qquad \forall a \in V, \ \forall n \in \mathbb{Z}$$

Therefore, up to constant multiples, one can draw:

$$\begin{array}{c}
Y(a,z) \\
\vdots \\
a_{(1)} \\
{\scriptstyle [\partial,-]}\downarrow \\
a_{(0)} \\
{\scriptstyle [\partial,-]}\downarrow \\
a_{(-1)} \\
{\scriptstyle [\partial,-]}\downarrow \\
a_{(-2)} \\
\vdots
\end{array}$$

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$$[\partial, a_{(n)}] = -n\, a_{(n-1)} \qquad \forall a \in V,\ \forall n \in \mathbb{Z}$$

Therefore, up to constant multiples, one can draw

if a vertex algebra has a supersymmetry D,

$$\begin{tikzcd}
Y(a,z) & Y(Da,z) \\
\vdots & \vdots \\
a_{(1)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & \\
 & (Da)_{(1)} \arrow[dl] \arrow[dd, "{[\partial,-]}"] \\
a_{(0)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & \\
 & (Da)_{(0)} \arrow[dl] \arrow[dd, "{[\partial,-]}"] \\
a_{(-1)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & \\
 & (Da)_{(-1)} \arrow[dl, "{[D,-]}"] \\
a_{(-2)} & \vdots \\
\vdots &
\end{tikzcd}$$

Recall that on a vertex algebra, we have

$$[\partial, a_{(n)}] = -n\, a_{(n-1)} \qquad \forall a \in V,\ \forall n \in \mathbb{Z}$$

Therefore, up to constant multiples, one can draw

if a vertex algebra has a supercommuting supersymmetry D, $\tilde{D}$.

$$\begin{tikzcd}
Y(a,z) & Y(Da,z) & Y(\tilde{D}a,z) & Y(D\tilde{D}a,z) \\
\vdots & \vdots & \vdots & \vdots \\
a_{(1)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & & (\tilde{D}a)_{(1)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & \\
& (Da)_{(1)} \arrow[dd, "{[\partial,-]}"] \arrow[dl] & & (D\tilde{D}a)_{(1)} \arrow[dd, "{[\partial,-]}"] \arrow[dl] \\
a_{(0)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & & (\tilde{D}a)_{(0)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & \\
& (Da)_{(0)} \arrow[dd, "{[\partial,-]}"] \arrow[dl] & & (D\tilde{D}a)_{(0)} \arrow[dl, "{[D,-]}"] \arrow[ddll, bend left, "{[\tilde{D},-]}"] \\
a_{(-1)} \arrow[dd, "{[\partial,-]}"'] \arrow[dr] & & (\tilde{D}a)_{(-1)} \arrow[ddll, bend left, "{[\tilde{D},-]}"] & \\
& (Da)_{(-1)} \arrow[dl, "{[D,-]}"] & & \vdots \\
a_{(-2)} & & \vdots & \\
\vdots & \vdots & &
\end{tikzcd}$$

The vertex algebra $V^k_{N=1}(\mathfrak{g})$ is an $N=1$ SUSY vertex algebra extending $V^k(\mathfrak{g})$.

<u>Definition</u> Let $\mathfrak{g}$ be a simple Lie superalgebra equipped with an even supersymmetric invariant bilinear form $(\cdot|\cdot)$.

Then, $V^k_{N=1}(\mathfrak{g})$ is a vertex algebra freely generated by

$$\{ \bar{a}_i , D\bar{a}_i \} \text{ for } \{a_i\} : \text{basis of } \mathfrak{g}$$

For $a \in \mathfrak{g}$, let $\bar{a}$ have a parity opposite to a.

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For $a \in \mathfrak{g}$, let $\bar{a}$ have a parity opposite to a.

The OPE relations between the generators are

$$(D\bar{a})(z)(D\bar{b})(w) \sim \frac{(D\overline{[a,b]})(w)}{z-w} + \frac{(k+h^\vee)(a|b)}{(z-w)^2}$$

$$\bar{a}(z)(D\bar{b})(w) \sim \frac{\overline{[a,b]}(w)}{z-w}$$

$$\bar{a}(z)\,\bar{b}(w) \sim \frac{(k+h^\vee)(a|b)}{z-w}$$

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Then, $V^k_{N=1}(\mathfrak{g})$ is a vertex algebra freely generated by

$$\{\bar{a}_i,\ D\bar{a}_i\} \text{ for } \{a_i\} : \text{basis of } \mathfrak{g}$$

For $a \in \mathfrak{g}$, let $\bar{a}$ have a parity opposite to a.

$V^k_{N=1}(\mathfrak{g}) =$

$\langle D\bar{a} \mid a \in \mathfrak{g} \rangle$	$\langle \bar{a} \mid a \in \mathfrak{g} \rangle$
$(D\bar{a})(z)(D\bar{b})(w) \sim \dfrac{(D\overline{[a,b]})(w)}{z-w} + \dfrac{(k+h^\vee)(a\|b)}{(z-w)^2}$	$\bar{a}(z)\bar{b}(w) \sim \dfrac{(k+h^\vee)(a\|b)}{z-w}$

$\cong$ $V^{k+h^\vee}(\mathfrak{g})$

Note that for $k \neq -h^\vee$,

$$V^k_{N=1}(\mathfrak{g}) \cong V^k(\mathfrak{g}) \otimes F(\mathfrak{g})$$

as vertex algebras.

$$W^k(\mathfrak{sl}_{n+1|n}) \cong \mathrm{Com}(V^{\ell+1}_{N=1}(\mathfrak{gl}_n), V^{\ell}_{N=1}(\mathfrak{sl}_{n+1}))$$

For any vertex algebra V and its subalgebra $U \subset V$, the coset is defined by

$$\begin{aligned}\mathrm{Com}(U, V) &= \{ v \in V \mid [Y(v,z), Y(u,w)] = 0 \text{ for all } u \in U \} \\ &= \{ v \in V \mid v_{(n)} u = 0 \text{ for all } u \in U \text{ and } n \in \mathbb{Z}_{\geq 0} \}\end{aligned}$$

Note that one can naturally regard that

$$\begin{aligned} V^{\ell+1}_{N=1}(\mathfrak{gl}_n) &\subset V^{\ell}_{N=1}(\mathfrak{sl}_{n+1}) \\ D\bar{a} &\longmapsto D\bar{a} \\ \bar{a} &\longmapsto \bar{a} \end{aligned}$$

The level difference occurrs by $h^\vee(\mathfrak{gl}_n) = n$ and $h^\vee(\mathfrak{sl}_{n+1}) = n+1$.

GKO construction and its supersymmetric analog

Virasoro algebra & GKO construction

Recall Virasoro vertex algebra of central charge $c \in \mathbb{C}$

$$\mathrm{Vir}^c = \langle L \rangle$$

$$L(z)L(w) \sim \frac{\partial_w L(w)}{z-w} + \frac{2L(w)}{(z-w)^2} + \frac{c}{6(z-w)^4}$$

In an affine vertex algebra $V^k(\mathfrak{g})$ for $k \neq -h^\vee$, one can find via Sugawara construction a Virasoro field

$$L^k_{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum_i :v^i v_i:$$

of central charge $\dfrac{k \cdot \mathrm{sdim}\,\mathfrak{g}}{k+h^\vee}$.

Proposition (Goddard-Kent-Olive (1985))

Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{p} \subset \mathfrak{g}$ be its reductive Lie subalg.

Then,

$$\mathrm{Com}(V^{\vec{k}}(\mathfrak{p}), V^{\vec{k}}(\mathfrak{g}))$$

has a Virasoro field given by $L_{\mathfrak{g}}^{\vec{k}} - L_{\mathfrak{p}}^{\vec{k}}$.

Remark. In particular, when $\mathfrak{g} = \mathfrak{sl}_n \oplus \mathfrak{sl}_n$, the GKO construction motivates the coset realization

$$W^k(\mathfrak{sl}_n) \cong \mathrm{Com}(V^{\ell+1}(\mathfrak{sl}_n), V^{\ell}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)),$$

where $\frac{1}{k+h^\vee} - \frac{1}{\ell+h^\vee} = 1$ for $h^\vee = n$.

N=1 Neveu-Schwarz vertex algebra

The N=1 supersymmetric analog of a Virasoro vertex algebra is an N=1 Neveu-Schwarz vertex algebra NS^c of central charge $c \in \mathbb{C}$.

$$NS^c = \langle L, \underbrace{G}_{\text{odd}} \rangle$$

$$L(z)L(w) \sim \frac{\partial_w L(w)}{z-w} + \frac{2L(w)}{(z-w)^2} + \frac{c}{6(z-w)^4}$$

$$L(z)G(w) \sim \frac{\partial_w G(w)}{z-w} + \frac{3}{2} \cdot \frac{G(w)}{(z-w)^2}$$

$$G(z)G(w) \sim \frac{2L(w)}{z-w} + \frac{c}{3(z-w)^3}$$

N=1 Neveu-Schwarz vertex algebra

The $N=1$ supersymmetric analog of a Virasoro vertex algebra is an $N=1$ Neveu-Schwarz vertex algebra NS^c of central charge $c \in \mathbb{C}$.

$$NS^c = \langle L, \underbrace{G}_{\text{odd}} \rangle$$

$$L(z)L(w) \sim \frac{\partial_w L(w)}{z-w} + \frac{2L(w)}{(z-w)^2} + \frac{c}{6(z-w)^4}$$

$$L(z)G(w) \sim \frac{\partial_w G(w)}{z-w} + \frac{3}{2} \cdot \frac{G(w)}{(z-w)^2}$$

$$G(z)G(w) \sim \frac{2L(w)}{z-w} + \frac{c}{3(z-w)^3}$$

In NS^c, $D := G_{(0)}$ is a supersymmetry

and $DG = 2L$

$$\begin{tikzcd}
G(z) & 2L(z) \\
\vdots & \vdots \\
G_{(1)} \arrow[d] \arrow[r] & 2L_{(1)} \arrow[d] \arrow[dl] \\
G_{(0)} \arrow[d] \arrow[r] & 2L_{(0)} \arrow[dl, "{[D,-]}"] \\
G_{(-1)} & \\
\vdots & \vdots
\end{tikzcd}$$

Def Let V be an $N=1$ SUSY vertex algebra with a supersymmetry D.

Then, an odd $G \in V$ is called an $N=1$ superconformal vector if

(i) $G_{(0)} = D$

(ii) G and $L = \frac{1}{2} DG$ generate $N=1$ Neveu-Schwarz vertex alg.

For convenience, we say that V has an $N=1$ structure with (L, G).

Proposition (Linshaw–S.–Suh (2025))

Assume that V has an $N=1$ structure with (L^V, G^V) and $A \subseteq V$ also has an $N=1$ structure with (L^A, G^A)

Suppose that

$$L^V_{(2)} L^A = 0, \qquad G^V_{(0)} G^A = 2L^A, \qquad G^V_{(1)} G^A = 0.$$

Then $\mathrm{Com}(A, V)$ has an $N=1$ structure with $(L^V - L^A, G^V - G^A)$.

In particular,

$$NS^c \curvearrowright \mathrm{Com}(A, V)$$

where $c = c_V - c_A$.

The supersymmetric affine vertex algebra $V^k_{N=1}(\mathfrak{g})$ for $k \neq -h^\vee$ has an $N=1$ superconformal vector, constructed in [Kac–Todorov (1985)]

$$G^k_{\mathfrak{g}} = \frac{1}{k+h^\vee} \sum_{i,j} (v_i | v_j) :\bar{v}^i (D\bar{v}^j): + \frac{1}{3(k+h^\vee)^2} \sum_{i,j,r} (-1)^{p(v_j)} (v_i | [v_j, v_r]) :\bar{v}^i \bar{v}^j \bar{v}^r:$$

of central charge $\dfrac{k \cdot \mathrm{sdim}\,\mathfrak{g}}{k+h^\vee} + \frac{1}{2}\mathrm{sdim}\,\mathfrak{g}$.

$\Rightarrow$ For a reductive Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$, we have

$$NS^c \curvearrowright \mathrm{Com}(V^{\bullet}_{N=1}(\mathfrak{p}), V^{\ell}_{N=1}(\mathfrak{g}))$$

by the adjoint action.

In particular,

$$NS^c \curvearrowright \mathrm{Com}(V^{\ell+1}_{N=1}(\mathfrak{gl}_n), V^{\ell}_{N=1}(\mathfrak{sl}_{n+1}))$$

where $c = \dfrac{3n\ell}{\ell+n+1}$.

N=2 Neveu-Schwarz vertex algebra

The N=2 supersymmetric analog of a Virasoro vertex algebra is an N=2 Neveu-Schwarz vertex algebra $NS^c_{N=2}$ of central charge $c \in \mathbb{C}$.

$$NS^c_{N=2} = \langle L, \underbrace{G}_{\text{odd}}, \underbrace{\tilde{G}}_{\text{odd}}, J \rangle$$

such that $\langle L, G \rangle \cong NS^c \cong \langle L, \tilde{G} \rangle$

$$J(z)J(w) \sim -\frac{c}{3} \cdot \frac{1}{(z-w)^2}$$

$$L(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}$$

$$G(z)J(w) \sim \frac{\tilde{G}(w)}{z-w}, \qquad \tilde{G}(z)J(w) \sim \frac{-G(w)}{z-w}$$

$$G(z)\tilde{G}(w) \sim \frac{2J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}$$

In $NS^c_{N=2}$, $D := G_{(0)}$ and $\tilde{D} := \tilde{G}_{(0)}$ makes $N=2$ SUSY so that

$$\begin{tikzcd}
 & 2L & \\
G \arrow[ur, "D"] \arrow[rr, leftrightarrow, "J_{(0)}"] & & \tilde{G} \arrow[ul, "\tilde{D}"'] \\
 & J \arrow[ul, "-\tilde{D}"] \arrow[ur, "D"'] &
\end{tikzcd}$$

In the specific form of the coset, we have $NS^c_{N=2}$ as a subalgebra.

Theorem (Kazama-Suzuki (1989), CLKSS (2026))

Let $\ell \neq -n-1$. Then

$$\mathrm{Com}(V^{\ell+1}_{N=1}(\mathfrak{gl}_n), V^{\ell}_{N=1}(\mathfrak{sl}_{n+1}))$$

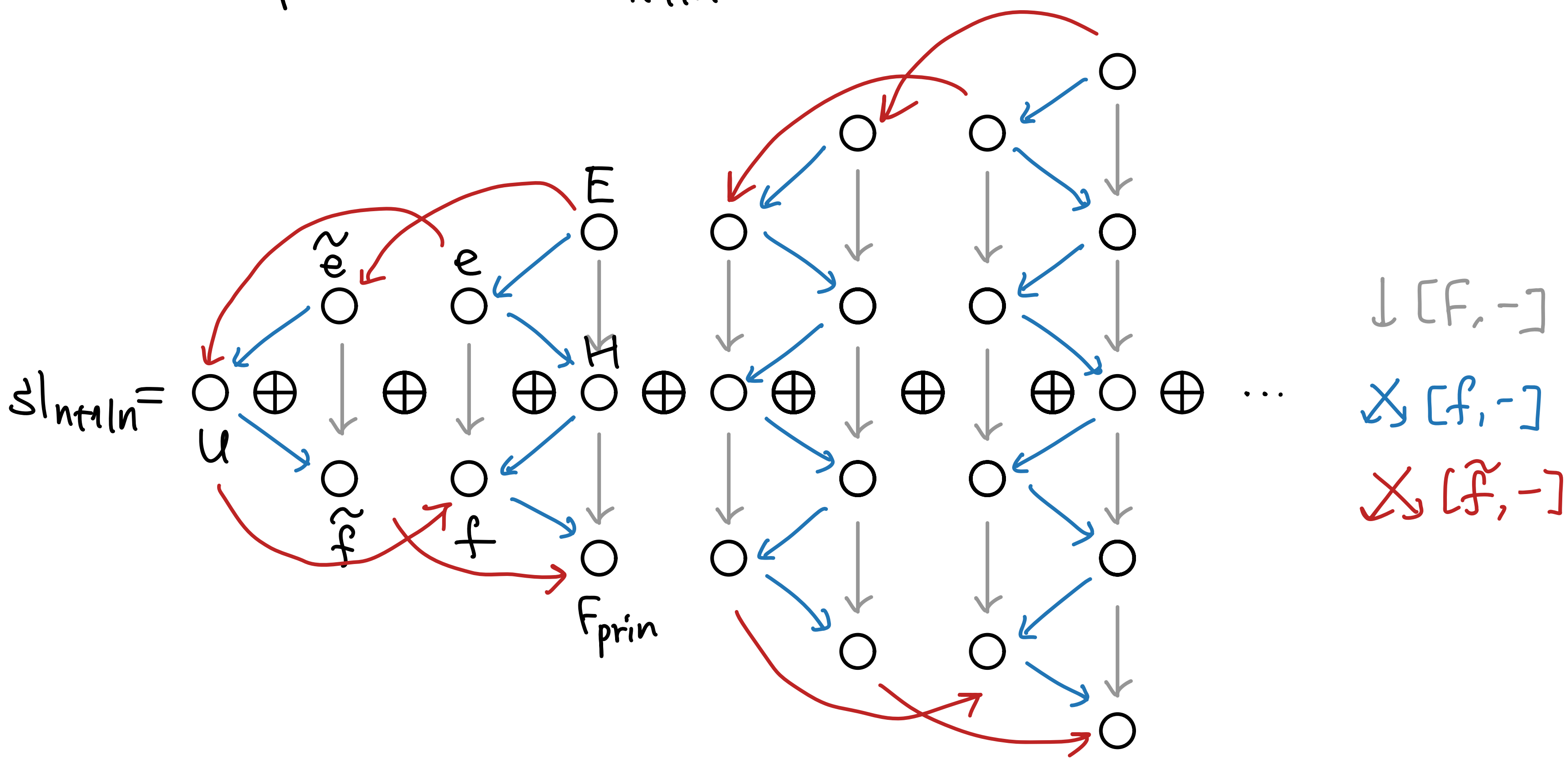
has a subalgebra isomorphic to $NS^c_{N=2}$, extending the $N=1$ NS vertex algebra given by the difference of the Kac-Todorov vectors.

Here, the central charge is

$$c = \frac{3n\ell}{\ell+n+1}$$

Connection with $W^k(\mathfrak{sl}_{n+1|n})$

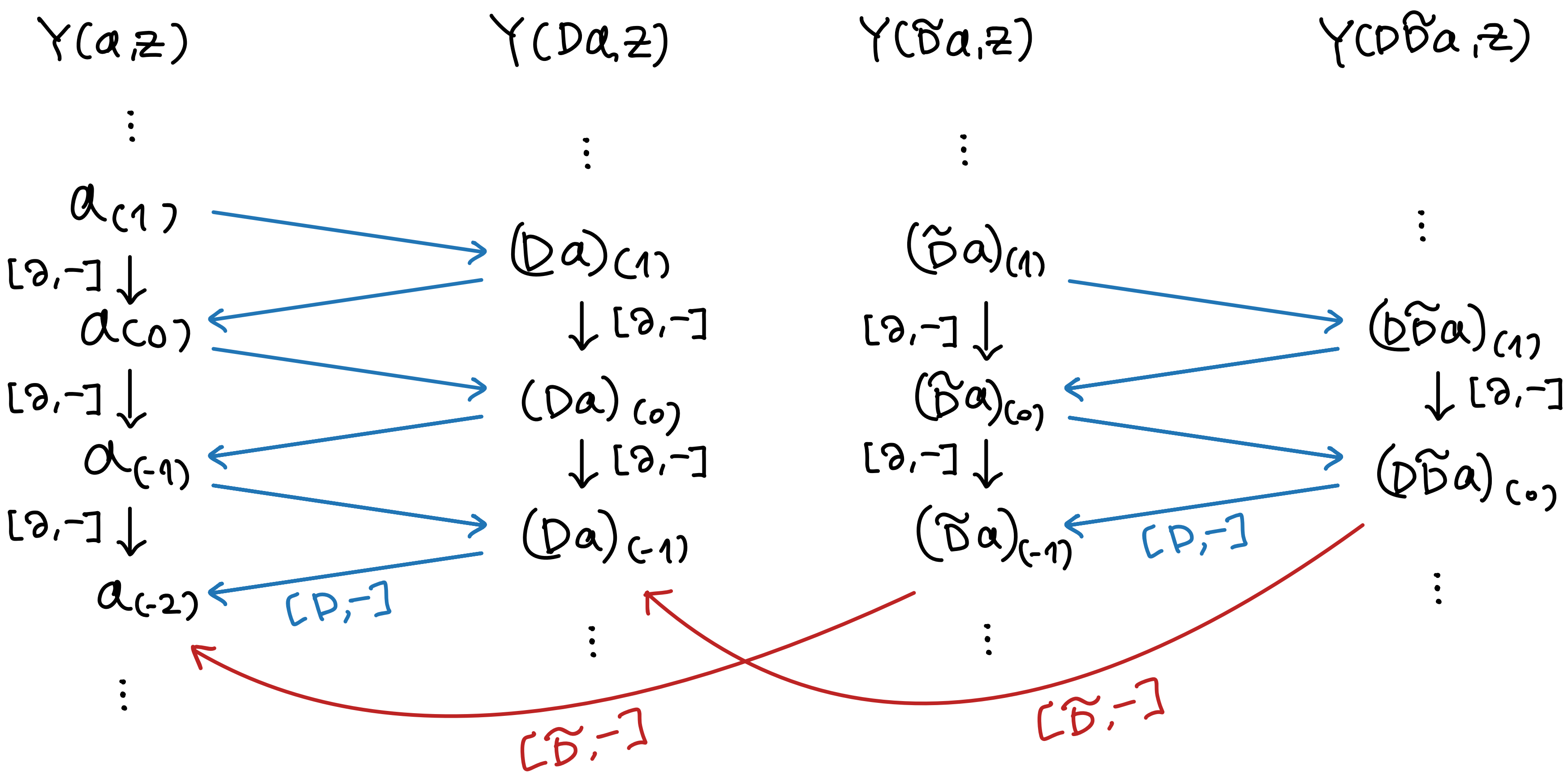
Recall the decomposition of $\mathfrak{sl}_{n+1|n}$



$[f,f] = 2F$, $[\tilde{f},\tilde{f}] = 2F$ $\leftsquigarrow$ $[D,D] = 2D^2 = 2\partial$

$[u,u] = 0$, $[f,\tilde{f}] = 0$

$[u,f] = \tilde{f}$, $[u,\tilde{f}] = -f$



$Y(a,z)$
$Y(Da,z)$
$Y(\tilde{D}a,z)$
$Y(D\tilde{D}a,z)$
$a_{(1)}$
$a_{(0)}$
$a_{(-1)}$
$a_{(-2)}$
$(Da)_{(1)}$
$(Da)_{(0)}$
$(Da)_{(-1)}$
$(\tilde{D}a)_{(1)}$
$(\tilde{D}a)_{(0)}$
$(\tilde{D}a)_{(-1)}$
$(D\tilde{D}a)_{(1)}$
$(D\tilde{D}a)_{(0)}$
$[\partial,-]$
$[D,-]$
$[\tilde{D},-]$

Theorem (Genra-S.-Suh (2025), Linshaw-S.-Suh (2025))

Let $\mathfrak{g}$ be a basic classical Lie superalgebra and $F \in \mathfrak{g}$ be an even nilpotent element.

Assume that there is an odd homogeneous $f \in \mathfrak{g}$ such that $[f, f] = 2F$.

Then for $k \neq -h^\vee$,

$$W^k(\mathfrak{g}, F) \otimes F(\mathfrak{g}^{\frac{1}{2}})$$

has an $N=1$ SUSY vertex algebra structure.

Moreover, it has an $N=1$ superconformal vector.

By applying an automorphism of $\mathfrak{sl}_{n+1|n}$ which exchanges f and $\tilde{f}$, we get the following theorem.

Theorem [CLKSS(2026)]

For $k \neq -1$, the W-algebra $W^k(\mathfrak{sl}_{n+1|n})$ has an $N=2$ SUSY vertex algebra structure. Moreover, it has a subalgebra isomorphic to

$$NS^c_{N=2}, \qquad c = -3n(kn + k + n).$$

Finally, by analyzing the $NS^c_{N=2}$-module structure and further vertex algebraic properties, we get

Theorem [CLKSS(2026)]

For $k \neq -1$,

$$W^k(\mathfrak{sl}_{n+1|n}) \cong \mathrm{Com}(V^{\ell+1}_{N=1}(\mathfrak{gl}_n),\ V^{\ell}_{N=1}(\mathfrak{sl}_{n+1}))$$

where $(k+1)(\ell+n+1) = 1$.

Thank you!