$W(2,2)$ tensor category

Gordan Radobolja

Faculty of Science, University of Split

Braiding new ground Maynhoot
Sep 18, 2026

Partially supported by the *Croatian Science Foundation* under the project IP-2022-10-9006.

Funding

This work was supported by the QuantiXLie Centre of Excellence, a project Co-financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004).

EUROPSKI STRUKTURNI I INVESTICIJSKI FONDOVI

Operativni program KONKURENTNOST I KOHEZIJA

Europska unija
Zajedno do fondova EU

Papers

Based on a recent joint work:

D. Adamović, M. Peng, J. Yang, *The tensor category for $W(2,2)$ vertex algebra*, (2026) arxiv:2609.02202

Papers

And some previous results:

- D.A., G.R. *On free field realization of* $W(2,2)$*-modules*, SIGMA 12 (2016)
- D.A., G.R. *Free field realization of the twisted Heisenberg–Virasoro algebra at level zero and its applications*, J. Pure Appl. Algebra 219 10 (2015)
- G.R. *Subsingular vectors in Verma modules, and tensor product of weight modules over the twisted Heisenberg–Virasoro algebra and* $W(2,2)$ *algebra*, J. Math. Phys. 54 (2013)

Generalities

Rational VOA: finite semi-simple category, MTC

C_2-cofinite VOA: finite logarithmic category, braided, rigid

Non-C_2-cofinite VOA: ??? (usually restrict to nicer sub-cat)

Huang:

grading-restricted + C_1-cofinite generalized modules

=

braided monoidal cat

abelian?

Generalities

Rational VOA: finite semi-simple category, MTC

C_2-cofinite VOA: finite logarithmic category, braided, rigid

Non-C_2-cofinite VOA: ??? (usually restrict to nicer sub-cat)

Huang:

$$\text{grading-restricted} \ + \ C_1\text{-cofinite} \ \text{generalized modules}$$
$$=$$
$$\text{braided monoidal cat}$$

abelian?

Generalities

Rational VOA: finite semi-simple category, MTC

C_2-cofinite VOA: finite logarithmic category, braided, rigid

Non-C_2-cofinite VOA: ??? (usually restrict to nicer sub-cat)

Huang:

$$\text{grading-restricted} \;+\; C_1\text{-cofinite} \;\; \text{generalized modules} \\ = \\ \text{braided monoidal cat}$$

abelian?

Strategy

Creutzig, Jiang, Orosz Huniker, Ridout, Yang (Virasoro), Adamović, Lin, Yang ($\mathbb{Z}_2$-orbifold of Heisenberg):

$\mathcal{C}$ grading-restricted C_1-cofinite generalized	$=$	$\mathcal{C}^{\text{fin}}$ finite-length with C_1-cofinite composition factors
$\Downarrow$		$\Downarrow$
braided monoidal		abelian

Fusion rules

Frenkel–Zhu theory H. Li refinement generator $L[2]$	+	detailed h.w. module structure
$\Downarrow$		$\Downarrow$
fusion boundaries		existence

$$L[r] \boxtimes L[s] \cong \bigoplus_{j=0}^{\min(r-1,s-1)} L[r+s-1-2j].$$

Etinghof, Penneys: $\Longrightarrow$ rigidity of $\mathcal{C}$

Fusion rules

Frenkel–Zhu theory H. Li refinement generator $L[2]$	$+$	detailed h.w. module structure
$\Downarrow$		$\Downarrow$
fusion boundaries		existence

$$L[r] \boxtimes L[s] \cong \bigoplus_{j=0}^{\min(r-1,s-1)} L[r+s-1-2j].$$

Etinghof, Penneys: $\Longrightarrow$ rigidity of $\mathcal{C}$

Fusion rules

Frenkel–Zhu theory H. Li refinement generator $L[2]$	$+$	detailed h.w. module structure
$\Downarrow$		$\Downarrow$
fusion boundaries		existence

$$L[r] \boxtimes L[s] \cong \bigoplus_{j=0}^{\min(r-1,s-1)} L[r+s-1-2j].$$

Etinghof, Penneys: $\implies$ rigidity of $\mathcal{C}$

Equivalence to cat of fin-dim $\mathfrak{sl}_2$-modules

$\mathcal{C}^{\mathrm{ss}}$
semisimple rigid subcategory built from $L[r]$, $r \in \mathbb{Z}_{>0}$

$\longleftrightarrow$

Rep $\mathfrak{sl}_2$
finite-dim $\mathfrak{sl}_2$-modules

$\Updownarrow$

$q=1$

$\mathcal{C}(q, \mathfrak{sl}_2)$
finite-dim weight mod of the quantum group $U_q(\mathfrak{sl}_2)$

(tensor equivalences)

Equivalence to cat of fin-dim $\mathfrak{sl}_2$-modules

$\mathcal{C}^{\mathrm{ss}}$
semisimple rigid subcategory built from $L[r]$, $r \in \mathbb{Z}_{>0}$

$\longleftrightarrow$

Rep $\mathfrak{sl}_2$
finite-dim $\mathfrak{sl}_2$-modules

$\Updownarrow$

$q=1$

$\mathcal{C}(q, \mathfrak{sl}_2)$
finite-dim weight mod of the quantum group $U_q(\mathfrak{sl}_2)$

(tensor equivalences)

$W(2,2)$ algebra

The $W(2,2)$ VOA $L_c^{\mathcal{W}}$, for $c \in \mathbb{C}^\times$ is a simple vertex-algebra generated by

$$\omega(z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}, \qquad W(z) = \sum_{n\in\mathbb{Z}} W(n) z^{-n-2}$$

such that $L(n)$, $W(n)$ satisfy relations of a $W(2,2)$ Lie algebra:

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n+m,0}\frac{n^3-n}{12}c$$
$$[L(n), W(m)] = (n-m)W(n+m) + \delta_{n+m,0}\frac{n^3-n}{12}c$$
$$[W(n), W(m)] = 0$$

Representations

The Verma module $V(c, h_L, h_W) = U(\mathcal{W}).v_{p,r}$

$$L(0)v_{p,r} = h_L v_{p,r}, \qquad W(0)v_{p,r} = h_W v_{p,r}$$

Parametrize: $V[p,r] := V(h_L[p,r], h_W[p,r])$

$$h_L = h_L[p,r] = (1-p^2)\frac{c-26}{24} + 1 - \frac{r+1}{2}p$$

$$h_W[p,r] = (1-p^2)\frac{c}{24}$$

Representations

The Verma module $V(c, h_L, h_W) = U(\mathcal{W}).v_{p,r}$

$$L(0)v_{p,r} = h_L v_{p,r}, \qquad W(0)v_{p,r} = h_W v_{p,r}$$

Parametrize: $V[p,r] := V(h_L[p,r], h_W[p,r])$

$$h_L = h_L[p,r] = (1-p^2)\frac{c-26}{24} + 1 - \frac{r+1}{2}p$$
$$h_W[p,r] = (1-p^2)\frac{c}{24}$$

Representations

Generic case $p \notin \mathbb{Z}_{>0}$

$V[p,r]$ is irreducible iff $p \notin \mathbb{Z}_{>0}$.

Non-generic case $p \in \mathbb{Z}_{>0}$

If $p \in \mathbb{Z}_{>0}$, there is a singular vector

$$u_p = U_p v_{p,r}, \qquad U_p \in \mathbb{C}[W(-1), \ldots, W(-p)]$$

such that $U(\mathcal{W})u_p \cong V[p, r-2]$.

U_p can be normalized at $W(-p)$.

Representations

Generic case $p \notin \mathbb{Z}_{>0}$

$V[p,r]$ is irreducible iff $p \notin \mathbb{Z}_{>0}$.

Non-generic case $p \in \mathbb{Z}_{>0}$

If $p \in \mathbb{Z}_{>0}$, there is a singular vector

$$u_p = U_p v_{p,r}, \qquad U_p \in \mathbb{C}[W(-1), \ldots, W(-p)]$$

such that $U(\mathcal{W})u_p \cong V[p, r-2]$.

U_p can be normalized at $W(-p)$.

Typical case $p \in \mathbb{Z}_{>0}$, $r \notin \mathbb{Z}_{>0}$

Typical case $r \notin \mathbb{Z}_{>0}$

If $r \notin \mathbb{Z}_{>0}$, then

$$0 \longrightarrow V[p, r-2] \longrightarrow V[p, r] \longrightarrow L[p, r] \longrightarrow 0$$

is exact.

Atypical case $p, r \in \mathbb{Z}_{>0}$

Atypical case $r \in \mathbb{Z}_{>0}$

If $r \in \mathbb{Z}_{>0}$, there is a subsingular vector

$$s_{p,r} = S_{p,r} v_{p,r}, \qquad \overline{V}[p,r] = U(\mathcal{W}) s_{p,r}$$

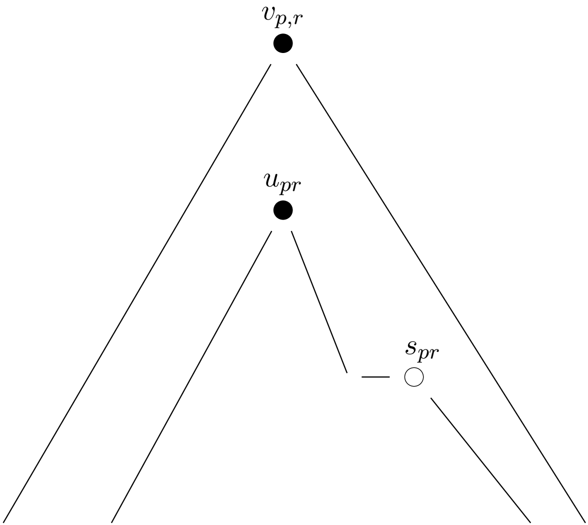
such that

$$0 \longrightarrow \overline{V}[p,r] \longrightarrow V[p,r] \longrightarrow L[p,r] \longrightarrow 0$$
$$0 \longrightarrow V[p,r-2] \longrightarrow \overline{V}[p,r] \longrightarrow L[p,-r] \longrightarrow 0$$

are exact.

$S_{p,r}$ can be normalized at $L(-p)^r$.

$V[p,r]$ atypical case



C_1-cofinite modules

Definition

Let M be a V-module, and

$$C_1(M) = \mathrm{span}_{\mathbb{C}} \left\{ v_{-1}m \mid m \in M,\ v \in \coprod_{n \in \mathbb{Z}_{\geq 1}} V_{(n)} \right\}.$$

M is C_1-cofinite if $\mathrm{codim}\,(M) < \infty$.

$$\frac{M[p,r]}{C_1(M[p,r])} = \mathrm{span}_{\mathbb{C}}\{W(-1)^w L(-1)^\ell v_{p,r} \mid w, \ell \in \mathbb{Z}_{\geq 0}\}$$

C_1-cofinite modules

Definition

Let M be a V-module, and

$$C_1(M) = \mathrm{span}_{\mathbb{C}} \left\{ v_{-1}m \mid m \in M,\ v \in \coprod_{n \in \mathbb{Z}_{\geq 1}} V_{(n)} \right\}.$$

M is C_1-cofinite if $\operatorname{codim}(M) < \infty$.

$$\frac{M[p,r]}{C_1(M[p,r])} = \mathrm{span}_{\mathbb{C}}\{W(-1)^w L(-1)^\ell v_{p,r} \mid w, \ell \in \mathbb{Z}_{\geq 0}\}$$

Main results

Theorem

Highest-weight modules in $\mathcal{C}$ are precisely

$$L[r] := L[1, r], \quad r \in \mathbb{Z}_{>0}.$$

Theorem ($\mathcal{C} = \mathcal{C}^{\mathrm{fin}}$)

If $M \in \mathcal{C}$, then it admits a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_{i+1}/M_i \cong L[r_i]$, $r_i \in \mathbb{Z}_{>0}$, and all M_i are C_1-cofinite.

Main results

Theorem

Highest-weight modules in $\mathcal{C}$ are precisely

$$L[r] := L[1, r], \quad r \in \mathbb{Z}_{>0}.$$

Theorem ($\mathcal{C} = \mathcal{C}^{\text{fin}}$)

If $M \in \mathcal{C}$, then it admits a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_{i+1}/M_i \cong L[r_i]$, $r_i \in \mathbb{Z}_{>0}$, and all M_i are C_1-cofinite.

The Frenkel–Zhu Correspondence

VOA Category	Zhu algebra category
VOA V	Associative Zhu algebra $A(V)$
V-module $M = \bigoplus_{n\geq 0} M(n)$	$A(V)$-bimodule $A(M)$
Intertwining operators $\mathcal{I}\binom{M_3}{M_1\ M_2}$	$\mathrm{Hom}_{A(V)}\big(A(M_1)\otimes_{A(V)} M_2(0),\, M_3(0)\big)$

H. Li

If M_2 and M_3' are generalized Verma modules

$$\mathcal{I}\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix} \cong \mathrm{Hom}_{A(V)}\big(A(M_1)\otimes_{A(V)} M_2(0),\, M_3(0)\big)$$

The Frenkel–Zhu Correspondence

VOA Category	Zhu algebra category
VOA V	Associative Zhu algebra $A(V)$
V-module $M = \bigoplus_{n\geq 0} M(n)$	$A(V)$-bimodule $A(M)$
Intertwining operators $\mathcal{I}\binom{M_3}{M_1\ M_2}$	$\mathrm{Hom}_{A(V)}\big(A(M_1)\otimes_{A(V)} M_2(0),\, M_3(0)\big)$

H. Li

If M_2 and M_3' are generalized Verma modules

$$\mathcal{I}\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix} \cong \mathrm{Hom}_{A(V)}\big(A(M_1)\otimes_{A(V)} M_2(0),\, M_3(0)\big)$$

Frenkel-Zhu theory for $L_c^{\mathcal{W}}$

- Zhu algebra: $A\left(L_c^{\mathcal{W}}\right) \cong \mathbb{C}[x, y]$
- M an $\mathbb{Z}_{>0}$-graded $L_c^{\mathcal{W}}$-module with top component $M(0)$
- $\mathbb{C}[x_1, x_2, y_1, y_2] \otimes M(0)$ is a $\mathbb{C}[x, y]$-bimodule
- $$A\left(V[p, r]\right) \cong \mathbb{C}[x_1, x_2, y_1, y_2] \otimes V[p, r](0)$$
- $$A\left(L[p, r]\right) = \frac{A\left(V[p, r]\right)}{A\left(\overline{V}[p, r]\right)} \cong \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{(f(U_{p,r}), g(S_{p,r}))}$$
- Calculate $A\left(L[2]\right)$

Frenkel-Zhu theory for $L_c^{\mathcal{W}}$

- Zhu algebra: $A\left(L_c^{\mathcal{W}}\right) \cong \mathbb{C}[x,y]$
- M an $\mathbb{Z}_{>0}$-graded $L_c^{\mathcal{W}}$-module with top component $M(0)$
- $\mathbb{C}[x_1,x_2,y_1,y_2] \otimes M(0)$ is a $\mathbb{C}[x,y]$-bimodule
- $$A\left(V[p,r]\right) \cong \mathbb{C}[x_1,x_2,y_1,y_2] \otimes V[p,r](0)$$
- $$A\left(L[p,r]\right) = \frac{A\left(V[p,r]\right)}{A\left(\overline{V}[p,r]\right)} \cong \frac{\mathbb{C}[x_1,x_2,y_1,y_2]}{(f(U_{p,r}),g(S_{p,r}))}$$
- Calculate $A\left(L[2]\right)$

Frenkel-Zhu theory for $L_c^{\mathcal{W}}$

- Zhu algebra: $A\left(L_c^{\mathcal{W}}\right) \cong \mathbb{C}[x,y]$
- M an $\mathbb{Z}_{>0}$-graded $L_c^{\mathcal{W}}$-module with top component $M(0)$
- $\mathbb{C}[x_1,x_2,y_1,y_2] \otimes M(0)$ is a $\mathbb{C}[x,y]$-bimodule
- $$A\left(V[p,r]\right) \cong \mathbb{C}[x_1,x_2,y_1,y_2] \otimes V[p,r](0)$$
- $$A\left(L[p,r]\right) = \frac{A\left(V[p,r]\right)}{A\left(\overline{V}[p,r]\right)} \cong \frac{\mathbb{C}[x_1,x_2,y_1,y_2]}{(f(U_{p,r}), g(S_{p,r}))}$$
- Calculate $A\left(L[2]\right)$

Theorem

For $p, p', r, r' \in \mathbb{C}$ we have

$$\dim I\begin{pmatrix} & L[p',r'] & \\ L[2] & & L[p,r]\end{pmatrix} \leq 1,$$

and this dimension can be nonzero only if $p' = p$ and for $p \neq 0$ if

$$r' = r \pm 1.$$

Existence

The existence of the intertwining operators of the type

$$\begin{pmatrix} & L[r \pm 1] & \\ L[2] & & L[r]\end{pmatrix}, \qquad r \in \mathbb{Z}_{>1}$$

can be proved either directly, or by embedding $L_c^{\mathcal{W}} \hookrightarrow L^{\mathcal{H}}_{c,c_{L,I}}$.

Theorem

For $p, p', r, r' \in \mathbb{C}$ *we have*

$$\dim I\begin{pmatrix} & L[p',r'] & \\ L[2] & & L[p,r]\end{pmatrix} \leq 1,$$

and this dimension can be nonzero only if $p' = p$ *and for* $p \neq 0$ *if*

$$r' = r \pm 1.$$

Existence

The existence of the intertwining operators of the type

$$\begin{pmatrix} & L[r\pm 1] & \\ L[2] & & L[r]\end{pmatrix}, \qquad r \in \mathbb{Z}_{>1}$$

can be proved either directly, or by embedding $L_c^{\mathcal{W}} \hookrightarrow L^{\mathcal{H}}_{c,c_{L,I}}$.

Main results

Theorem

We have the following fusion rules in $\mathcal{C}$:

$$L[r] \boxtimes L[s] \cong \bigoplus_{j=0}^{\min(r-1,s-1)} L[r+s-1-2j].$$

Theorem

The tensor category $\mathcal{C}$ is rigid.

Etinghof, Penneys: non-negligible + moderate growth = rigid

Main results

Theorem

We have the following fusion rules in $\mathcal{C}$:

$$L[r] \boxtimes L[s] \cong \bigoplus_{j=0}^{\min(r-1,s-1)} L[r+s-1-2j].$$

Theorem

The tensor category $\mathcal{C}$ is rigid.

Etinghof, Penneys: non-negligible + moderate growth = rigid

Tensor equivalence $\mathcal{C}^{\mathrm{ss}} \cong \mathcal{C}(1, \mathfrak{sl}_2) \cong \operatorname{Rep} \mathfrak{sl}_2$

Rep $\mathfrak{sl}_2$ finite-dim $\mathfrak{sl}_2$-modules	$\Leftrightarrow$	$\mathcal{C}(q, \mathfrak{sl}_2)$ finite-dim weight mod of the quantum group $U_q(\mathfrak{sl}_2)$

generated by $L_q(1)$

Kazhdan, Wenzl:

rigid ss tensor cat simple generator X Clebsch-Gordan rule	$\Leftrightarrow$ $\mathcal{C}(q, \mathfrak{sl}_2)$, $X \leftrightarrow L_q(1)$

$q = 1$

Tensor equivalence $\mathcal{C}^{\mathrm{ss}} \cong \mathcal{C}(1, \mathfrak{sl}_2) \cong \operatorname{Rep} \mathfrak{sl}_2$

$\operatorname{Rep} \mathfrak{sl}_2$ finite-dim $\mathfrak{sl}_2$-modules	$\Leftrightarrow$	$\mathcal{C}(q, \mathfrak{sl}_2)$ finite-dim weight mod of the quantum group $U_q(\mathfrak{sl}_2)$

generated by $L_q(1)$

Kazhdan, Wenzl:

rigid ss tensor cat simple generator X Clebsch-Gordan rule	$\Leftrightarrow$	$\mathcal{C}(q, \mathfrak{sl}_2),\ X \leftrightarrow L_q(1)$

$q = 1$

Tensor equivalence $\mathcal{C}^{ss} \cong \mathcal{C}(1, \mathfrak{sl}_2) \cong \operatorname{Rep} \mathfrak{sl}_2$

Rep $\mathfrak{sl}_2$ finite-dim $\mathfrak{sl}_2$-modules	$\Leftrightarrow$	$\mathcal{C}(q, \mathfrak{sl}_2)$ finite-dim weight mod of the quantum group $U_q(\mathfrak{sl}_2)$

generated by $L_q(1)$

Kazhdan, Wenzl:

rigid ss tensor cat simple generator X Clebsch-Gordan rule	$\Leftrightarrow$	$\mathcal{C}(q, \mathfrak{sl}_2),\ X \leftrightarrow L_q(1)$

$$q = 1$$