More on Grothendieck–Verdier categories

Max Demirdilek and Christoph Schweigert

Mathematics Department
Hamburg University

partially based on work with
Jürgen Fuchs, Christian Reiher, Gregor Schaumann and Simon Wood

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I. Two tensor products

Recap: GV categories

Definition

Grothendieck–Verdier category: monoidal category $(\mathcal{C}, \otimes)$ with

- Dualizing object $K \in \mathcal{C}$.
- Duality functor: antiequivalence $G : \mathcal{C} \to \mathcal{C}^{opp}$ such that

$$\mathrm{Hom}(X \otimes Y, K) \cong \mathrm{Hom}(X, GY)\,.$$

Choice of K is structure.

Theorem (Allen 2023, D 2026)

Let H be a Hopf algebroid with finite-dimensional base algebra A. If H has an invertible antipode S (Allen) or the vector space dual A^* of the base algebra A carries an H-module structure (Demirdilek), then the category of finite-dimensional H-modules is a GV category with dualizing object A^*.

GV categories and rigidity

Remark

Symmetric GV categories are known since the seventies as $*$-autonomous categories and correspond to linearly distributive categories with negation.

Definition

1. The right dual of an object c in a monoidal category $\mathcal{C}$ is an object $c^\vee \in \mathcal{C}$, together with evaluation and coevaluation morphisms

$$\mathrm{ev}^r : c^\vee \otimes c \to 1 \qquad \mathrm{coev}^r : 1 \to c \otimes c^\vee$$

   such that the usual zig-zag relations hold.
2. A left dual is defined analogously.
3. A category is called rigid if all its objects have both a left and right dual.

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Examples (of rigid categories)

1. Finite-dimensional vector spaces; finite-dimensional modules over a finite-dimensional Hopf algebra over a field.
2. Category of tangles, cobordisms.

Consequences of rigidity

Consequences of rigidity:

- The tensor product of an abelian rigid tensor category is exact.
- Projective objects form a monoidal ideal.
- The dual is opmonoidal,

$$(x \otimes y)^\vee \cong y^\vee \otimes x^\vee \, .$$

Finite-dimensional bimodules over a finite-dimensional k-algebra carry a structure of a GV category which we will now study.

Bimodules revisited

k a field, A a finite-dimensional k-algebra.
(All k-vector spaces will be finite-dimensional.)

Facts

1. If B is an A-bimodule, its linear dual $B^* = \mathrm{Hom}_k(B, k)$ with action

$$(a_2.\beta.a_1)(b) := \beta(a_1.b.a_2)$$

is an A-bimodule $G(B)$. In particular, A^* is an A-bimodule.

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2. Example: $A = k[x, y]/(x^2, y^2, xy)$ with one-dimensional simple module S

$$0 \to S \oplus S \to A \to S \to 0$$

   shows that A^* is, in general, not isomorphic to A.

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3. One has

$$\mathrm{Hom}_{A\text{-bimod}}(M_1 \otimes_A M_2, A^*) \cong \mathrm{Hom}_{A\text{-bimod}}(M_1, (M_2)^*) \,.$$

So A^* is a dualizing object for A-bimod.

A second tensor product for bimodules

Facts

- The tensor product $B \otimes_A \tilde{B}$ of two bimodules B and $\tilde{B}$ is a coequalizer

$$B \otimes A \otimes \tilde{B} \overset{\rightarrow}{\rightarrow} B \otimes \tilde{B} \rightarrow B \otimes_A \tilde{B} \rightarrow 0$$

  and thus right exact. Monoidal unit is A.

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- A^* is a coalgebra, any $M \in A\text{-bimod}$ is a bicomodule. Equalizer

$$0 \to B \otimes^A \tilde{B} \to B \otimes \tilde{B} \rightrightarrows B \otimes A^* \otimes \tilde{B}$$

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  gives left exact tensor product with monoidal unit A^*.
- The two tensor products are related by duality

$$B \otimes^A \tilde{B} \cong G(G(\tilde{B}) \otimes_A G(B))$$

  which is not opmonoidal but rather relates the two tensor products.

The second tensor product of bimodules

The second tensor product $\otimes^A$ is very useful:

Facts

- Eilenberg–Watts:
  $F : A_1\text{-mod} \to A_2\text{-mod}$ right exact, then $F(-) \cong F(A_1) \otimes_{A_1} -$
  $H : A_1\text{-mod} \to A_2\text{-mod}$ left exact, then $H(-) \cong H(A_1^*) \otimes^{A_1} -$
- Morphism of algebras $A \to A'$ $\rightsquigarrow$ restriction $A'\text{-mod} \to A\text{-mod}$
  Left adjoint = Induction is $A' \otimes_A -$
  Right adjoint = Coinduction is $(A')^* \otimes^A -$

Properties of GV categories

Definition

Internal Hom: $\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$.
Internal coHom: $\mathrm{Hom}(X, Y \otimes Z) \cong \mathrm{Hom}(\underline{\mathrm{coHom}}(Z, X), Y)$.

Remarks

1. A second tensor product

$$X \parr Y := G(G^{-1}Y \otimes G^{-1}X)$$

   with the dualizing object K as a monoidal unit has internal coHoms and is thus continuous.

2. In a GV category, internal Homs

$$\underline{\mathrm{Hom}}(X, Z) \cong G(X \otimes G^{-1}Z) \cong Z \parr G(X)$$

   exist. Thus $\otimes$ is cocontinuous.

3. Trivializing the double dual $G^2 \rightsquigarrow$ Notion of pivotal GV category.
4. Notions of braided GV category, ribbon GV category exist.

GV categories and quantum topology

Theorem (Müller–Woike 2020-2025)

1. *Cyclic associative algebras in* Lex^f $\leftrightarrow$ *pivotal GV categories*
2. *Cyclic framed* E_2*-algebras in* Lex^f $\leftrightarrow$ *ribbon GV categories.*
3. *Cyclic framed* E_2*-algebra* $\leftrightarrow$ *ansular functor.*
4. *Cyclic associative algebra* $\leftrightarrow$ *open modular functor.*

Theorem (Brochier–Woike 2022)

Under a genus one condition, the ansular functor uniquely extends to a modular functor. This condition holds for modular categories, but also the Feigin–Fuchs boson or Drinfeld centers of pivotal finite tensor categories with non-trivial dualizing object.

II. Linear logic and graphical calculus

- In logic, we are used to two connectives AND and OR.
- The definition of the second tensor product

$$X \parr Y := G(G^{-1}Y \otimes G^{-1}X)$$

reminds us of de Morgan's law.

Classical logic

- Classical logic allows assumptions to be freely copied or discarded: from the statement:

  $\boxed{\text{I am in Maynooth.}} \Longrightarrow \boxed{\text{I am in Ireland.}}$

  we can derive the statement

  $\boxed{\text{I am in Maynooth.}} \Longrightarrow \boxed{\text{I am in Ireland.}} \wedge \boxed{\text{I am in Maynooth.}}$

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- From a proof $X \to Y$ we obtain a proof $X \to X$ and Y.
- In other settings, copying becomes problematic: from the statement

  $\boxed{\text{I have 3 Euros.}} \implies \boxed{\text{I can buy a large salad.}}$

  we cannot derive the statement

  $\boxed{\text{I have 3 Euros.}} \implies \boxed{\text{I have 3 Euros.}} \wedge \boxed{\text{I can buy a large salad.}}$

A quick (categorical) look at logic

Variables:	$X, Y, Z, \ldots$
Connectives:	$\otimes$ as well $\parr$
Formulas:	$(X \otimes Y) \parr (X \otimes Z)$, $((X \parr X) \parr X) \otimes Z$,

A quick (categorical) look at logic

Variables: $X, Y, Z, \ldots$
Connectives: $\otimes$ as well $\invamp$
Formulas: $(X \otimes Y) \invamp (X \otimes Z)$, $((X \invamp X) \invamp X) \otimes Z$,

Arrows between formulas represent: processes, programs, proofs

$$X \to Y.$$

Arrows can be composed:

$$X \ \to Y, \ \ Y \to Z, \qquad\qquad X \xrightarrow{\text{composite}} Z.$$

We have identity arrows (trivial program):

$$X \xrightarrow{\mathrm{id}_X} X.$$

We do not talk about negation at the moment.

Associators

Some canonical proofs/programs should exist, encoding e.g. the associativity of **AND**/**OR** in our framework.

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$$X \parr (Y \parr Z) \xrightarrow{\overline{\alpha}} (X \parr Y) \parr Z$$

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How does $\otimes$ interact with $\parr$?

$$X \otimes (Y \parr Z) \xleftrightarrow{?} (X \otimes Y) \parr (X \otimes Z)$$
$$(X \parr Y) \otimes Z \xleftrightarrow{?} (X \otimes Z) \parr (Y \otimes Z)$$

Naive distributors are *not* resource-sensitive: they require a doubling.

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How does $\otimes$ interact with $\parr$?

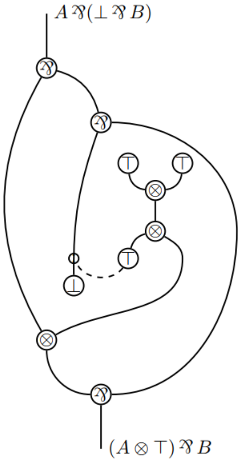
$$X \otimes (Y \parr Z) \xrightarrow{\delta^l} (X \otimes Y) \parr Z$$

$$(X \parr Y) \otimes Z \xrightarrow{\delta^r} X \parr (Y \otimes Z)$$

There is a price to pay: These arrows are only one-way, but still obey pentagon (and triangle) relations: Linearly distributive category.

Graphical calculus

First idea: draw proof nets



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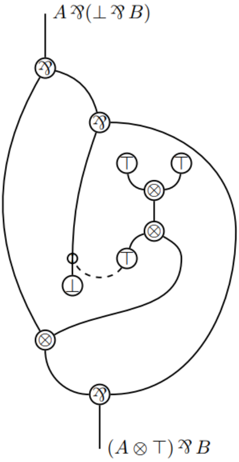

Needs non-local correctness criteria like Girard's long trip criterion.

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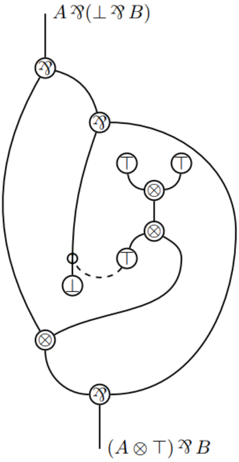

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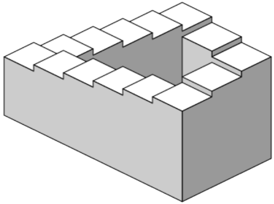
Idea: the structure is really three-dimensional, the criteria ensure that the two-dimensional graph is a valid projection.



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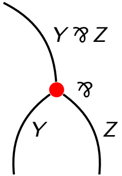

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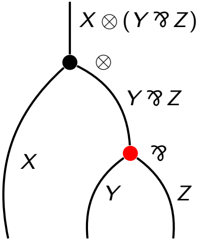

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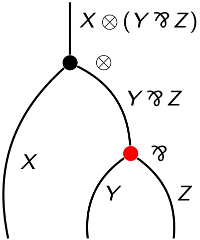

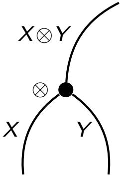

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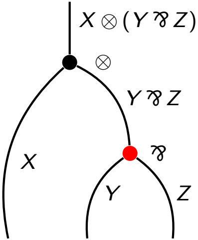

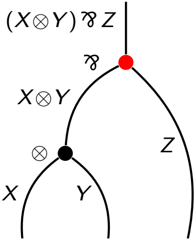

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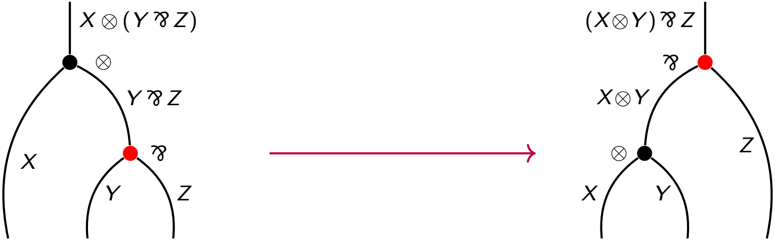

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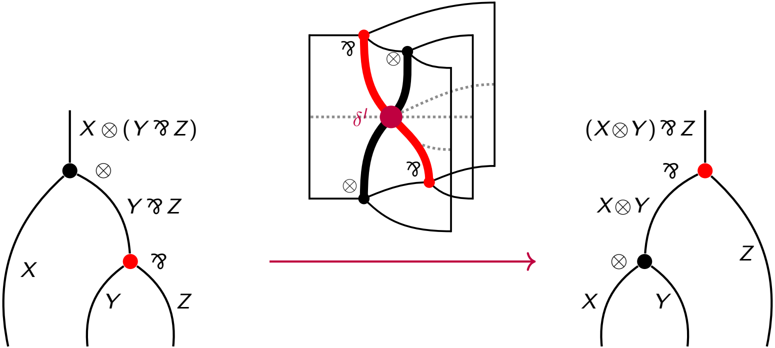

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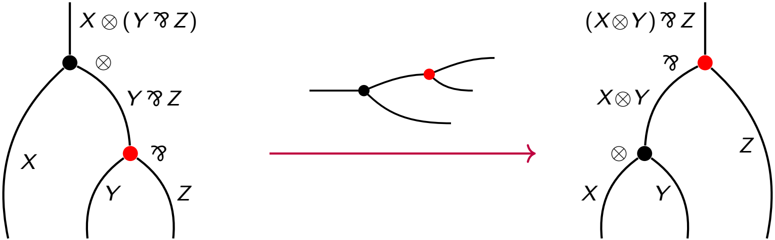

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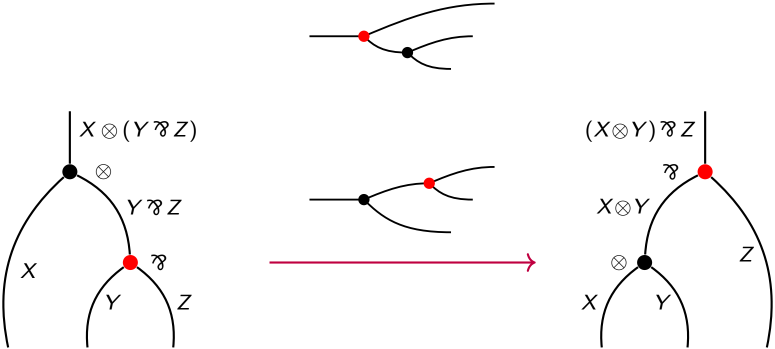

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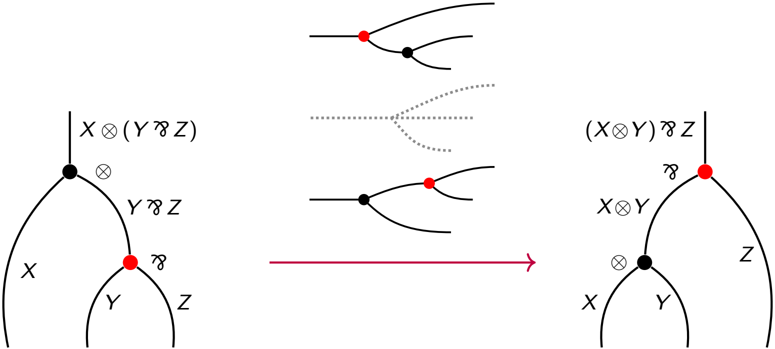

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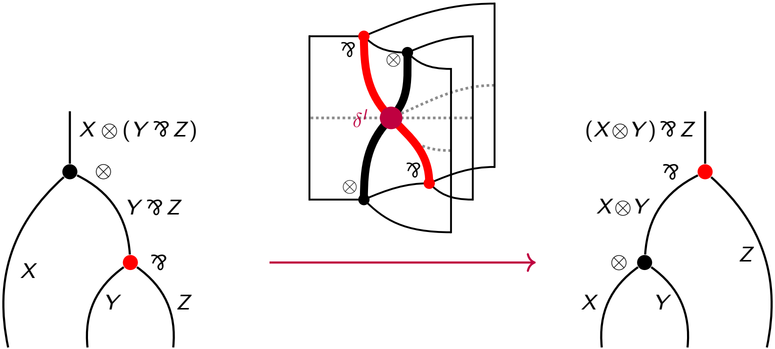
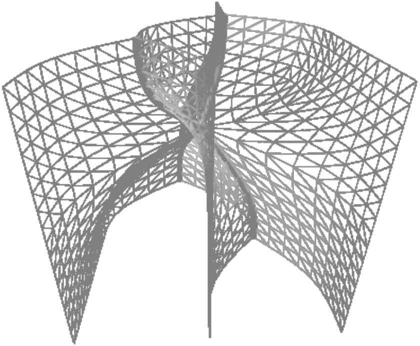


The distributor



A dictionary

Linear Logic	*2D Chiral CFT*
formulas	representations of the vertex operator algebra
arrows	intertwiners between representations
$\otimes$	coupling of representations (fusion product)
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The boundary fields fixing a specific boundary condition transform in a specific representation A. This object A should be some kind of Frobenius algebra.

III. More about distributors and module categories

Motivations:

- Full CFT needs module categories.
- Full CFT needs Frobenius algebras obtained from module categories to describe field objects.
- The two tensor products are on the same footing, we need to understand their relation, i.e. distributors

$$\delta^l : \; c_1 \otimes (c_2 \parr c_3) \to (c_1 \otimes c_2) \parr c_3 \quad \text{and} \quad \delta^r : (c_1 \parr c_2) \otimes c_3 \to c_1 \parr (c_2 \otimes c_3)\,.$$

GV module categories

Definition

A left GV module category over a GV category $(\mathcal{C}, \otimes, K)$ is a left module category $(\mathcal{M}, \triangleright)$ over $(\mathcal{C}, \otimes)$ such that all functors

$$- \triangleright m : {}_{\mathcal{C}}\mathcal{C} \to \mathcal{M} \text{ with } c \mapsto c \triangleright m \quad \text{and} \quad c \triangleright - : \mathcal{M} \to \mathcal{M} \text{ with } m \mapsto c \triangleright m$$

admit a right adjoint.

Call $R_c : \mathcal{M} \to \mathcal{M}$ the right adjoint to $c \triangleright -$.

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Proposition (Fuchs, Schaumann, S, Wood)

Let $(\mathcal{M}, \triangleright)$ be a left GV module category over a GV category $(\mathcal{C}, \otimes)$. Then

$$\blacktriangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M} \quad \textit{with} \quad c \blacktriangleright m := R_{Gc}(m)$$

defines a (left exact) left module category structure over $(\mathcal{C}, ⅋)$.

$\Rightarrow$ Existence of inner Homs for the action $\triangleright$ and inner coHoms for the action $\blacktriangleright$.

Lax module functors

Fact

$\mathcal{C}$ k-linear monoidal category, $\mathcal{M}, \mathcal{N}$ left $\mathcal{C}$-modules

1. Linear functor $F\colon \mathcal{M} \to \mathcal{N}$ with $G\colon \mathcal{N} \to \mathcal{M}$ right adjoint.
   Canonical bijection
   Oplax $\mathcal{C}$-module functor structures on $F \leftrightarrow$ Lax $\mathcal{C}$-module structures on G
   satisfying a condition that is formulated in terms of module profunctors.
2. F strong module functor $\Rightarrow$ G unique structure of a lax $\mathcal{C}$-module functor

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3. Apply to the module functor $c \mapsto c \rhd m$
   $\Rightarrow$ for all $m \in \mathcal{M}$ the functor $\underline{\mathrm{Hom}}(m, -)$ is a lax module functor
   $\Rightarrow$ coherent morphisms

$$\delta^l : c \otimes \underline{\mathrm{Hom}}(m, n) \to \underline{\mathrm{Hom}}(m, c \rhd n) \qquad \text{for } c \in \mathcal{C} \text{ and } m, n \in \mathcal{M} \; .$$

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4. Special case: for $\mathcal{M} = \mathcal{C}$ obtain distributors
   $$\delta^l : c \otimes \underline{\mathrm{Hom}}(x,y) = c \otimes (y \parr G(x)) \to (c \otimes y) \parr G(x) = \underline{\mathrm{Hom}}(x, c \otimes y)$$
   obeying pentagon (and triangle) diagrams.

Bimodules leading to strong module functors

Lemma

Let A be a finite-dimensional k-algebra, and let ${}_AM_A \in A\text{-}\mathrm{bimod}$ be a finite-dimensional A-bimodule.

The following statements are equivalent:

(i) $\underline{\mathrm{Hom}}(M,-)$ *is a strong module functor.*

(ii) M *has an* $\otimes_A$*-right rigid dual.*

(iii) M_A *is projective as a right A-module.*

(iv) *For all* $X, Y \in A\text{-}\mathrm{bimod}$ *the distributor* $\delta^l\colon\ X \otimes_A (Y \otimes^A M^*) \to\ (X \otimes_A Y) \otimes^A M^*$ *is an isomorphism.*

Important subcategories

Definition (Fuchs, Schaumann, S, Wood)

1. An object $m \in \mathcal{M}$ is called $\otimes$-admissible if
    1. The functor $\underline{\mathrm{Hom}}(m, -)$ is a strong $\triangleright$- module functor.
    2. The functor $\underline{\mathrm{Hom}}(m, -)$ has a right adjoint.
2. $\invamp$-admissible objects are defined using $\underline{\mathrm{coHom}}$.
3. $\widehat{\mathcal{M}}^{\otimes} / \widehat{\mathcal{M}}^{\invamp}$:= full subcategories of $\mathcal{M}$ of $\otimes$/ $\invamp$ -admissible objects.
4. The subcategories $\widehat{\mathcal{C}}^{\otimes} / \widehat{\mathcal{C}}^{\invamp}$ of $\mathcal{C}$ are obtained by considering $\mathcal{C}$ as a left GV module category over itself.

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   2. The functor $\underline{\mathrm{Hom}}(m,-)$ has a right adjoint.
2. $⅋$-admissible objects are defined using $\underline{\mathrm{coHom}}$.
3. $\widehat{\mathcal{M}}^{\otimes} / \widehat{\mathcal{M}}^{⅋}$:= full subcategories of $\mathcal{M}$ of $\otimes$/ $⅋$ -admissible objects.
4. The subcategories $\widehat{\mathcal{C}}^{\otimes} / \widehat{\mathcal{C}}^{⅋}$ of $\mathcal{C}$ are obtained by considering $\mathcal{C}$ as a left GV module category over itself.

Proposition (Fuchs, Schaumann, S, Wood)

1. *Let $\mathcal{C}$ be a GV category. The subcategories $\widehat{\mathcal{C}}^{\otimes}$ and $\widehat{\mathcal{C}}^{⅋}$ of $\mathcal{C}$ are non-unital monoidal subcategories.*
2. *Let $\mathcal{M}$ be a left GV module category over $\mathcal{C}$. By restriction, the category $\widehat{\mathcal{M}}^{\otimes}$ is a left $\widehat{\mathcal{C}}^{\otimes}$-module category and $\widehat{\mathcal{M}}^{⅋}$ is a left $\widehat{\mathcal{C}}^{⅋}$-module category.*

Partially defined relative Serre functors

Proposition

There exists an equivalence $S_{\mathcal{M}} \colon \widehat{\mathcal{M}}^{\otimes} \to \widehat{\mathcal{M}}^{\parr}$ *called the relative Serre functor such that*

$$\underline{\mathrm{Hom}}(m, -) \cong \underline{\mathrm{coHom}}(S_{\mathcal{M}}(m), -)$$

as an equivalence of $\otimes$*- and* $\parr$*-module functors.*

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as an equivalence of $\otimes$- and $\parr$-module functors.

Proposition

Let $\mathcal{C}$ be a GV category and $\mathcal{M}$ a left GV module category over $\mathcal{C}$.

1. *The relative Serre functor of $\mathcal{C}$ is canonically a monoidal equivalence*

$$S_{\mathcal{C}} : \widehat{\mathcal{C}}^{\otimes} \to \widehat{\mathcal{C}}^{\parr}$$

2. *The relative Serre functor $S_{\mathcal{M}}$ of $\mathcal{M}$ is a twisted module functor,*

$$S_{\mathcal{M}}(c \triangleright m) \cong S_{\mathcal{C}}(c) \blacktriangleright S_{\mathcal{M}}(m)$$

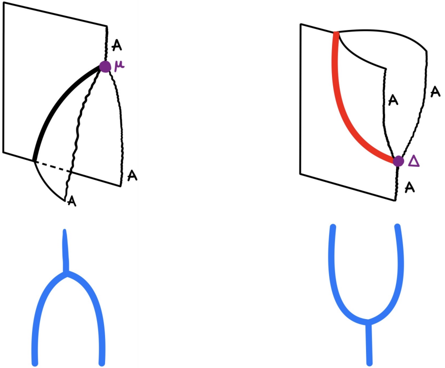
   for $c \in \widehat{\mathcal{C}}^{\otimes}$ and $m \in \widehat{\mathcal{M}}^{\otimes}$.

IV. Frobenius algebras

For field objects in full CFTs, we need Frobenius algebras.

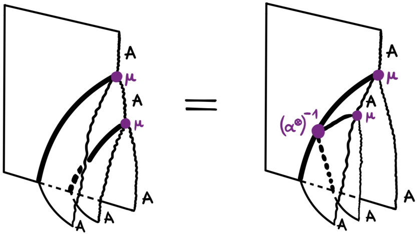
Frobenius algebras

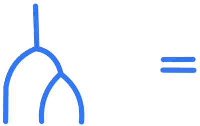
There exists a natural notion of a GV Frobenius algebra A.
It is defined simultaneously as an $\otimes$-algebra and as a $\parr$-coalgebra:



"The operator product of fields and the induced coproduct.
The operator product encodes crucial information about correlators."

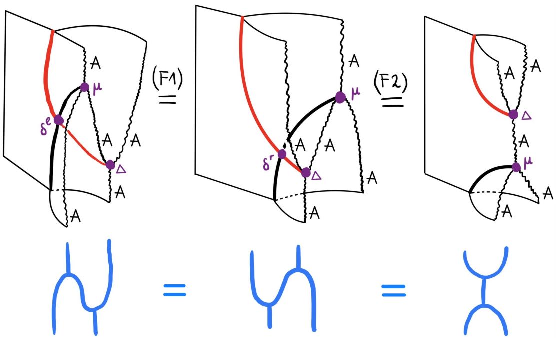
Associativity





"Associativity is the crucial consistency requirement on OPEs."
(Belavin–Polyakov–Zamolodchikov '84)

Mixed Associativity: Frobenius algebras



"Non-degeneracy of the two-point correlators."

Theorem (Fuchs, Schaumann, S, Wood)

Let $m \in \widehat{\mathcal{M}}^{\otimes}$. For every choice of isomorphism $p\colon m \to S_{\mathcal{M}}(m)$ in $\mathcal{M}$, $\underline{\mathrm{Hom}}(m,m)$ is a GV Frobenius algebra in $\mathcal{C}$ with Frobenius form

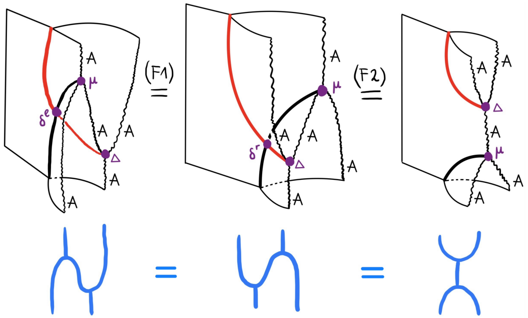
$$\lambda:\ \underline{\mathrm{Hom}}(m,m) \xrightarrow{\underline{\mathrm{Hom}}(m,p)} \underline{\mathrm{Hom}}(m, S_{\mathcal{M}}(m)) \xrightarrow{\mathrm{tr}_m} K\,.$$

Graphical computations

Frobenius algebras allow for graphical computations. For example:

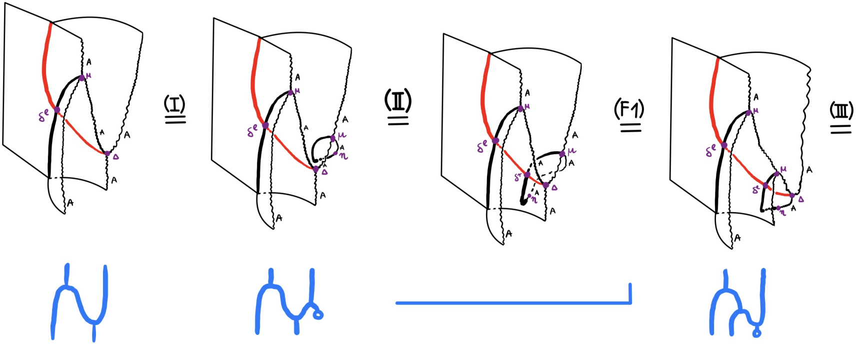
Proposition (D–S)

Relation (F1) implies (F2).



Proof

"The linear distributors work behind the scenes."



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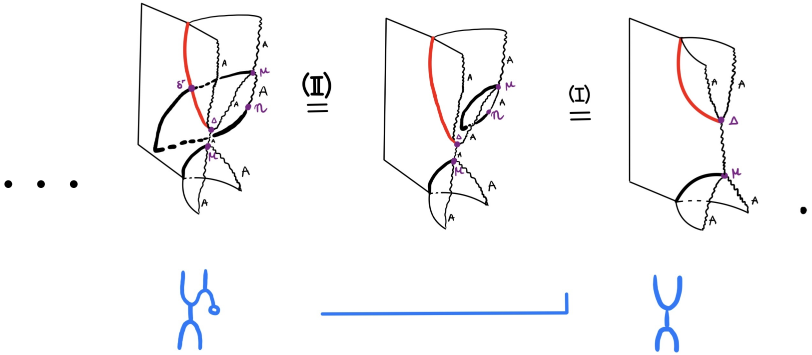

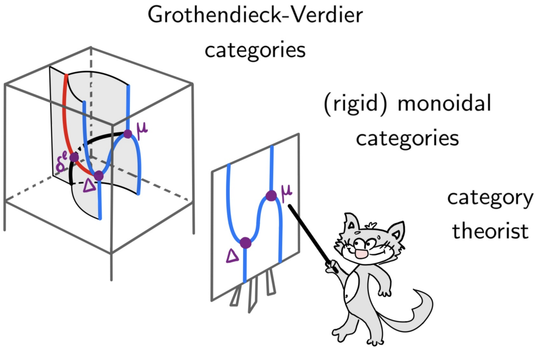
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Grothendieck-Verdier
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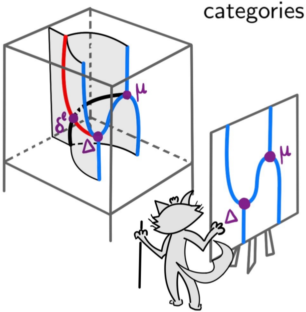

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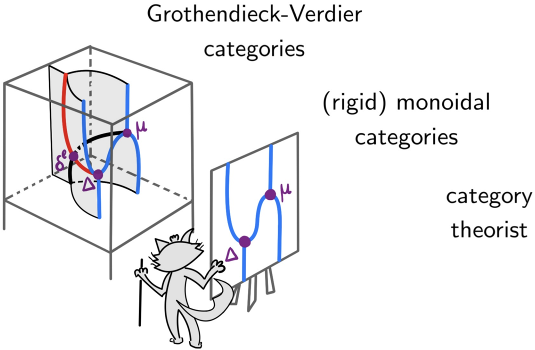

‘Coherence morphisms live in a third dimension.’

Coherence theorem

We prefer working with 2D rather than 3D diagrams.
The following *coherence theorem* makes this possible:

Theorem (D–Reiher–S, Došen–Petrić)

Any formal diagram in a linearly distributive category without units commutes.

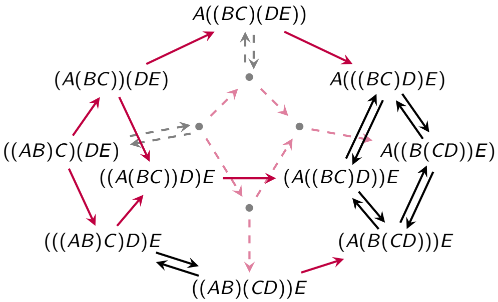

Figure: Directed associahedron for unbracketed $A \parr B \otimes C \otimes D \parr E$.

Developing a two-dimensional graphical calculus is work in progress.

V. Grothendieck–Verdier functors

What is a good notion of functor between Grothendieck–Verdier categories?

Grothendieck–Verdier functors

Let $\mathcal{C}, \mathcal{D}$ be closed monoidal categories, $K \in \mathcal{C}$, $k \in \mathcal{D}$, $F\colon \mathcal{C} \to \mathcal{D}$ a lax monoidal functor, and $\epsilon^F : F(K) \to k$.

For $X \in \mathcal{C}$, consider

$$\xi^r_X\colon\ F(\underline{\mathrm{Hom}}^{\mathrm{r}}(X,K)) \to \underline{\mathrm{Hom}}^{\mathrm{r}}(FX,FK) \xrightarrow{\underline{\mathrm{Hom}}^{\mathrm{r}}(FX,\epsilon^F)} \underline{\mathrm{Hom}}^{\mathrm{r}}(FX,k)\,.$$

$$\xi^l_X\colon\ F(\underline{\mathrm{Hom}}^{\mathrm{l}}(X,K)) \to \underline{\mathrm{Hom}}^{\mathrm{l}}(FX,FK) \xrightarrow{\underline{\mathrm{Hom}}^{\mathrm{l}}(FX,\epsilon^F)} \underline{\mathrm{Hom}}^{\mathrm{l}}(FX,k)\,.$$

Definition (D)

- ϵ^F is a Frobenius form if ξ^r_X and ξ^l_X are invertible for all $X \in \mathcal{C}$.
- If K and k are dualizing and ϵ^F is a Frobenius form, then (F, ϵ^F) is a Grothendieck–Verdier (GV) functor.

Remark

GV functors preserve GV Frobenius algebras.

Grothendieck–Verdier functors

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For $X \in \mathcal{C}$, consider

$$\xi^r_X\colon\ FG(X) \to \underline{\mathrm{Hom}}^{\mathrm{r}}(FX, FK) \xrightarrow{\underline{\mathrm{Hom}}^{\mathrm{r}}(FX,\epsilon^F)} GF(X)\,.$$

$$\xi^l_X\colon\ FG'(X) \to \underline{\mathrm{Hom}}^{\mathrm{l}}(FX, FK) \xrightarrow{\underline{\mathrm{Hom}}^{\mathrm{l}}(FX,\epsilon^F)} G'F(X)\,.$$

Definition (D)

- ϵ^F is a Frobenius form if ξ^r_X and ξ^l_X are invertible for all $X \in \mathcal{C}$.
- If K and k are dualizing and ϵ^F is a Frobenius form, then (F, ϵ^F) is a Grothendieck–Verdier (GV) functor.

Remark

GV functors preserve GV Frobenius algebras.

Lifting theorem

Theorem (D)

$$\begin{array}{ll} (\mathcal{C}, K) & \textit{closed monoidal with distinguished object} \\ \big\downarrow & F \ \textit{lax monoidal, iso-reflecting functor} \\ (\mathcal{D}, k) & \textit{GV category} \end{array}$$

If $\epsilon^F : F(K) \to k$ *Frobenius form* $\implies$ $(\mathcal{C}, K)$ *GV cat and* (F, ϵ^F) *GV functor.*

Applications

- H a (suitable) Hopf algebroid with finite-dimensional base A. Consider

  $$F\colon\ H\text{-mod} \longrightarrow A\text{-bimod}\,.$$

  $\Rightarrow$ $(H\text{-mod}, A^*)$ is GV (cf. Allen).
- A a commutative algebra in a (suitable) braided GV category $\mathcal{C}$. Consider

  $$F\colon\ {}_A\mathcal{C} \longrightarrow \mathcal{C} \quad \text{and} \quad F\colon\ {}_A\mathcal{C}^{\mathrm{loc}} \longrightarrow \mathcal{C}\,.$$

  $\Rightarrow$ $({}_A\mathcal{C}, GA)$ and $({}_A\mathcal{C}^{\mathrm{loc}}, GA)$ are GV (Creutzig–McRae–Shimizu–Yadav).

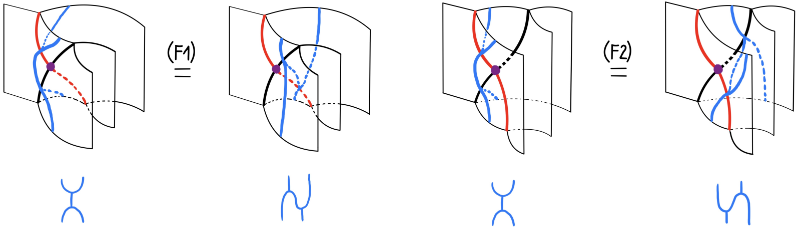
Frobenius linearly distributive functors

Definition (Cockett–Seely 1997)

Frobenius linearly distributive functor between linearly distributive categories:

- lax $\otimes$-monoidal structure,
- oplax ⅋-monoidal structure,

satisfying two *Frobenius relations*.



A correspondence

GV categories are linearly distributive (LD) categories with negation:

A correspondence

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Theorem (D)

$$\left\{ \begin{array}{c} \textit{LD categories with negation} \\ \textit{Frobenius LD functors} \\ + \textit{2-cells} \end{array} \right\} \quad \xleftrightarrow[\simeq]{\textit{2-equivalence}} \quad \left\{ \begin{array}{c} \textit{GV categories} \\ \textit{GV functors} \\ + \textit{2-cells} \end{array} \right\}$$

This theorem extends results of Cockett–Seely 1997 and Fuchs–Schaumann–S–Wood 2024.

Summary and outlook

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Grothendieck–Verdier duality is a natural structure:

- It appears in natural representation theoretic examples.
- It admits a theory of module categories, Serre functors, Frobenius algebras.

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Outlook

- Relation to modular functors, in particular using skein-theoretic methods.
- CFT correlators.