"NIM-REPRESENTATIONS OF TAMBARA-YAMAGAMI GENERALIZATIONS"

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@ BRAIDING NEW GROUND

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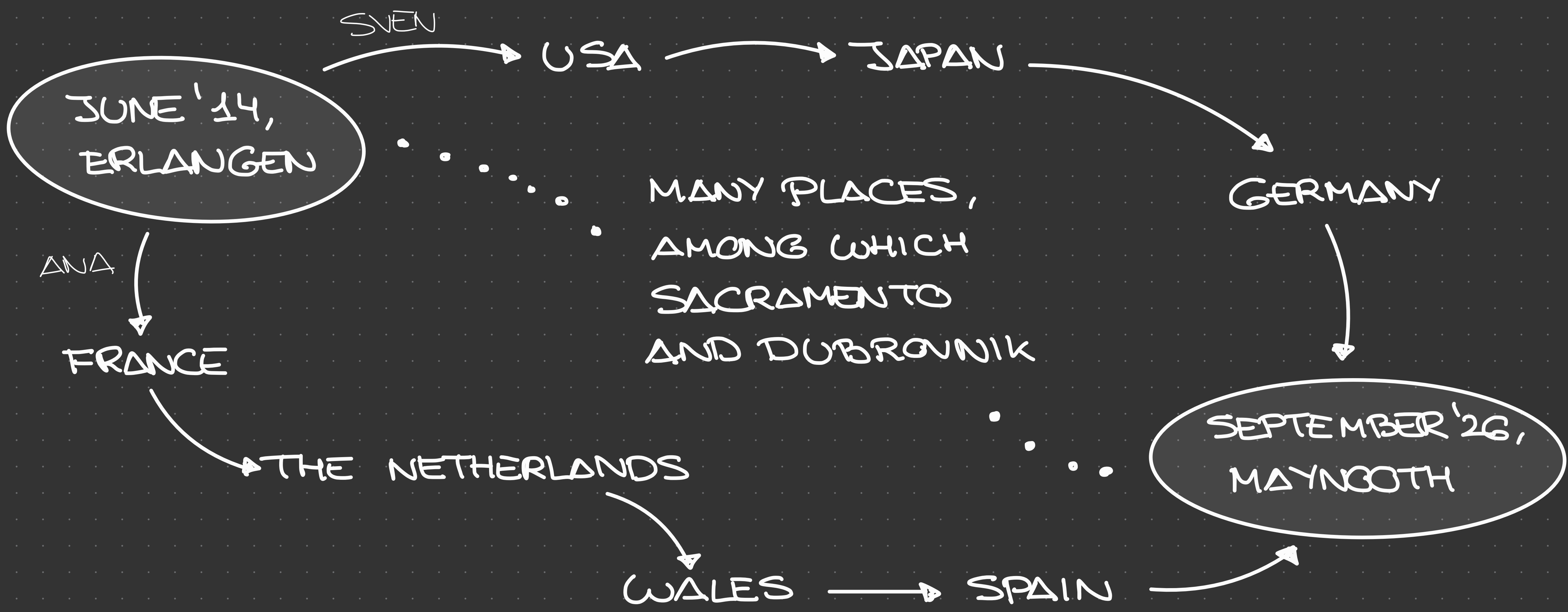
But first...

CONGRATULATIONS SVEN !!!

(AND ARNE)



Onto more serious business...

0 MOTIVATION

TAMBARA - YAMAGAMI CATEGORIES → Important in Math Phys!

Related to
- (2+1) TQFTs,
- $\mathbb{Z}_2$-orbifolds of conformal nets
- Etc.

Fusion Ring:
- basis: $G \cup \{X\}$, where G finite, abelian group and X non-invertible
- relations: $g \cdot X = X = X \cdot g$

$$X \cdot X = \bigoplus_{g \in G} g$$

Two proposals for generalization:

[JL] := [Jordan-Larsson, '09]
- G square order
- Get $\mathbb{Z}_p$-extensions of Vec_G $(p \in \mathbb{N})$
- Recover TY for $p = 2$

[Galindo-Lentner-Möller, '24] =: [GLM]

Party boy here!

- G abelian, even or odd order
- Get non-degenerate braided $\mathbb{Z}_2$-crossed extensions of Vec_G
- Recover TY with odd order

OUR QUESTION: if categorifiable, how does each suggestion change the representation theory of the categories?

OUR IDEA: (A. Czenky, E. Mc Govern, M. Mölander, M. Müller, ARC as a "Women in Mathematical Physics 3" research team)

Use Non-negative Integer Matrix representations ("NIM-reps") of these fusion rings to explore them!

WHY?

Because of previous work [Hannah-RC, '23], we knew NIM-reps are powerful combinatorial tools to detect algebra objects in tensor categories.
And by [Ostrik], we know these are directly related to the rep. theory of tensor categories.

... So off we went ☺

PLAN FOR TODAY:

1. Basics on NIM-reps.
2. Irreducible NIM-reps for [JL].
3. Irreducible NIM-reps for [GLM]
4. Bonus: algebra objects
5. Outro.

① BASICS ON NIM-REPS

NOTATION: $\mathbb{Z}_+$ semi-ring of positive integers with 0.
(R,B) fusion ring, where R ring and $B=\{b_i\}_{i\in I}$ basis with 1,
I finite set with an involution $*$.

DEFN: given (R,B), a NIM-rep is a module over R, say A, with a basis M that interacts with B as follows:
$\forall\, b_i\in B,\ m_j\in M,\ b_i \triangleright m_j = \sum_k a_{i,j}^k m_k$ with $a_{i,j}^k \in \mathbb{Z}_+$,
and that, if A has a symmetric bilinear form defined by
$(m_k, m_\ell) = \delta_{k,\ell}$,
then $(b_i \triangleright m_k, m_\ell) = (m_k, b_{i^*} \triangleright m_\ell)$.
Notation: (A,M).

Ex: a fusion ring is a NIM-rep over itself trivially.
(More examples to come)

Viewed as R-modules, we can define naturally direct sums of NIM-reps, sub-NIM-reps, etc.

DEFN:
- Irreducible NIM-reps: no proper sub-NIM-reps.
- Let (A,M), (A',M') NIM-reps over (R,B). A NIM-reps morphism is a function $\varphi: M \longrightarrow M'$ inducing a $\mathbb{Z}$-linear map between the resp. modules.
  If φ bijection $\Rightarrow$ NIM-rep isomorphism.

Ex: Let G finite group, and consider $(\mathbb{Z}G, G)$.

[Hannah-RC,'23]: irreducible NIM-reps of $(\mathbb{Z}G, G)$ correspond to transitive group actions of G.

$\Rightarrow$ basis elements are parametrized by left cosets of H in G, for some $H < G$,

i.e. $M = \{m_{g_i}\}_{1 \le i \le |G:H|}$

Isomorphism if H, K are conjugate subgroups.

Let us also introduce a visual way to describe NIM-reps:

for a given NIM-rep (A, M) over (R, B), a NIM-graph is constructed by drawing

- a node for each element of the basis M,
- a directed arrow with source m_i and target m_j, labeled by an element $b_k \in B$ for every copy of m_j in $b_k \triangleright m_i$.

Ex: for $(\mathbb{Z}G, G)$, the NIM-graph is the Cayley graph of G.

USEFUL BECAUSE... NIM-rep irreducible $\longleftrightarrow$ NIM-graph connected.

We will need a small variation of NIM-graphs, called NIM-orbit graph.

Given a NIM-graph, the NIM-orbit graph is given by contracting all edges labeled by elements of any (sub) groups parametrising the NIM-rep.

Ex: (in a second!)

(2) IRREDUCIBLE NIM-REPS OF [JL]

Let G finite group of square order, and $p>2$ positive integer.

[JL] fusion ring:
- basis: $G \cup \{X_1, \dots, X_{p-1}\}$
- relations: $g \cdot X_i = X_i = X_i \cdot g$,

$$X_i \cdot X_j = \begin{cases} \sqrt{|G|}\, X_{i+j}, & \text{if } i+j \neq p, \\ \sum_{g \in G} g, & \text{if } i+j = p. \end{cases}$$

involution: $X_i^* = X_{p-i}$.

Notation: $R_{p,G}$

Thanks to the group action, we can partition M into G-orbits:

$$M = \bigsqcup_{i=1}^{m} \{m_e^i\}_{1 \leq e \leq |G:H_i|}$$

where i accounts for the subgroups $H_i < G$ defining each orbit.

Then,

PROP: a) For a fixed orbit $1 \leq i \leq p$, and fixed $1 \leq k \leq p-1$, the summands in $X_k \triangleright m_e^i$ are contained in a unique orbit.

b) In any irreducible NIM-rep of $R_{p,G}$, the action of G partitions M into at most p orbits.

c) Let m be the number of orbits in the G-action. Then m divides p.

Rmk: we can identify X_1 with a permutation $\sigma \in S_m$.
In particular, if $m>1$ and $\mathcal{O} := \{\mathcal{O}_1, \dots, \mathcal{O}_m\}$ denotes the set of orbits, then we can visualize the action of X_1 on $\mathcal{O}$ as:

$$\begin{tikzcd}
\mathcal{O}_1 \arrow[r, "X_1"] & \mathcal{O}_2 \arrow[r, "X_1"] & \mathcal{O}_3 \arrow[r, "X_1"] & \dots \arrow[r, "X_1"] & \mathcal{O}_m \arrow[llll, bend left, "X_2"]
\end{tikzcd}$$

First example of a NIM-orbit graph!

For the 1-orbit case,

Prop: irreducible NIM-reps over $R_{p,G}$ with a unique orbit are classified by conjugacy classes of subgroups $H<G$ s.t. $\sqrt{|G|}$ divides $|H|$.

For the $m>1$ case,

Prop. Let $p>2$, $m \mid p$. Then,

i) Irreducible NIM-reps over $R_{p,G}$ with m orbits are parametrized by subgroups $H_1, \dots, H_m < G$ s.t. $c_i^k := |H_i||H_{\sigma^k(i)}| / |G|$ is a square integer $\forall i,k$.

ii) Any two such NIM-reps with associated subgroups $H_1, \dots, H_m$ and $L_1, \dots, L_m$ and constants a_i^k, b_i^k resp., are isomorphic $\Leftrightarrow$ $\exists \tau \in S_m$ s.t. H_i is conjugate to $L_{\sigma(i)}$ and $a_i^k = b_{\sigma(i)}^k$ $\forall i$.

Ex: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$
$p=4$

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For the $m > 1$ case,

THM: Let $p > 2$, $m \mid p$. Then,

i) Irreducible NIM-reps over $R_{p,G}$ with m orbits are parametrized by subgroups $H_1, \dots, H_m < G$ s.t. $c_i^k := |H_i||H_{\sigma^k(i)}| / |G|$ is a square integer $\forall i, k$.

ii) Any two such NIM-reps with associated subgroups $H_1, \dots, H_m$ and $L_1, \dots, L_m$ and constants a_i^k, b_i^k resp., are isomorphic $\iff \exists \tau \in S_m$ s.t. H_i is conjugate to $L_{\sigma(i)}$ and $a_i^k = b_{\sigma(i)}^k$ $\forall i$.

Ex: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$
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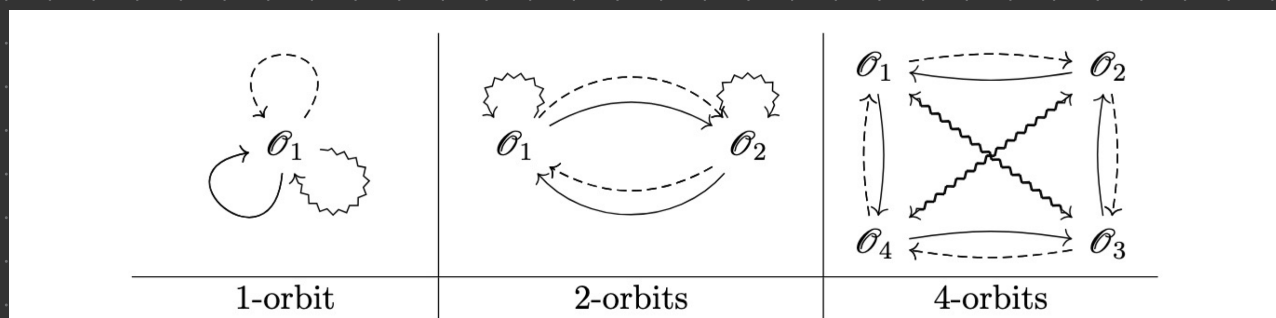

TABLE 1. $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $p = 4$, NIM-orbit graphs. Dashed arrows indicate action with X_1, resp. squiggle arrows with X_2 and plain arrows with X_3.

③ IRREDUCIBLE NIM-REPS OF [GLM]

Let Γ finite abelian group of even order, $\delta \in \Gamma/2\Gamma$.

[GLM] fusion ring: - basis: $\Gamma \cup \{X_{\bar{g}}\}_{\bar{g} \in \Gamma/2\Gamma}$

- relations:

$$g \cdot X_{\bar{h}} = X_{\bar{h}} \cdot g = X_{\bar{g}+\bar{h}}$$

$$X_{\bar{g}} \cdot X_{\bar{h}} = \sum_{\substack{t \in \Gamma \\ \bar{t} = \delta + \bar{g} + \bar{h}}} t$$

- involution: $(X_{\bar{g}})^* = X_{-\bar{g}-\delta}$

Notation: $GLM(\Gamma, \delta)$

We partition again M

1) into Γ-orbits: $M = \bigsqcup_{i=1}^{m} \mathcal{M}_j$

where j accounts for the distinct Γ-orbits, each of them defined by a stabilizer subgroup $H_j < \Gamma$.

2) into 2Γ-orbits: $\mathcal{M}_j = \bigsqcup_{i=1}^{N_j} \mathcal{O}_i^j$, where $\mathcal{O}_i^j := \{m_e^{i,j}\}_{1 \le e \le |2\Gamma : H_i^j|}$,

$H_i^j := H_j \cap 2\Gamma$,

$N_j := [\Gamma \cap 2\Gamma] / [H_i : H_i \cap 2\Gamma]$

This is a power of 2!

NOTATION: • to refer to the i-th 2Γ-orbit of the j-th Γ-orbit, we will call it an orbit pair and denote as (i,j).

• $c^{\bar{g}}_{(i,j),(k,l)} := (X_{\bar{g}} \triangleright m^{i,j}_r, m^{k,l}_s)$.

LEMMA: for an orbit pair (i,j) and $\bar{g} \in \Gamma/2\Gamma$,
1) $\exists\ (k,l)$ orbit pair such that $c^{\bar{g}}_{(i,j),(k,l)} \neq 0$
2) The summands in $X_{\bar{g}} \triangleright m^{i,j}_r$ are contained in a unique orbit pair.

But unlike [JL],

PROP: there are at most two Γ-orbits!

Also, the $\sigma \Leftrightarrow X_1$ identification cannot be done here. :(

Rather, for a given NIM-rep over $GLM(\Gamma, \delta)$,

a) If $\delta = 0$, then $X_{\bar{h}}$ acts on the 2Γ-orbits as a permutation of order 1 or 2,
b) If $\delta \neq 0$, then (...) 1, 2 or 4.

Not exactly what we were expecting, but then...

For the 1 orbit case,

THM: • Irreducible NIM-reps over $GLM(\Gamma, \delta)$ with a unique Γ-orbit are parametrized by

i) A choice of subgroup $H < \Gamma$ s.t. $\sqrt{|2\Gamma|}$ divides $|H \cap 2\Gamma|$, and

ii) A choice of action on the 2Γ-orbits induced by $X_{\bar{0}}$, that we denote as τ_0, satisfying that $\tau_0^2 = \sigma_\delta$

$$\tau_0 \sigma_g = \sigma_g \tau_0 \quad \forall g \in \Gamma.$$

• Isomorphism if:

i) $H = H'$,

ii) If τ is the isomorphism induced on the 2Γ-orbits by the isomorphism of Γ-sets, s.t. $\tau(\mathcal{O}_i) = \mathcal{O}'_{\tau(i)}$, then τ_0, τ_0' satisfy $\tau \cdot \tau_0 = \tau_0' \cdot \tau$.

For the 2 orbits case,

THM: • Irreducible NIM-reps over $GLM(\Gamma, \delta)$ with two Γ-orbits are parametrized by

i) A choice of subgroups $H_1, H_2 < \Gamma$ such that

a) $\delta = \bar{h}$ for some $h \in H_1 \cap H_2$, or $\delta = \bar{h}$ for $h \notin H_1, h \notin H_2$.

b) If we denote: $G[2] := \{ g \in G \mid 2g = 0 \}$, then

$$|(\Gamma/H_1)[2]| = |(\Gamma/H_2)|[2]$$

ii) A choice of τ_0 (as in the one orbit case), that also needs to satisfy some extra conditions on the number of 2Γ-orbits.

iii) An integer $\sqrt{\dfrac{|H_1 \cap 2\Gamma|\,|H_2 \cap 2\Gamma|}{|2\Gamma|}}$.

- Isomorphism under analogous conditions to the 1-orbit case.

4) Bonus: Algebra objects.

From [Hannah-RC, '23], we know there is a way to extract algebra objects from NIM-reps (if these rings are categorified).

$\longrightarrow$ Visually: look for loops at the NIM-(orbit) graphs!

We then detect:

[JL]

- 1 orbit case: $A = \bigoplus_{h\in H} b_h \bigoplus_{k} c_{1,1}^{k} X_k$
- m orbits case: $A_i = \bigoplus_{h\in H_i} b_h \bigoplus_{j=1}^{e-1} c_i^{jm} X_{jm}$, for $p = me$.

[GLM]

- 1 orbit case: $A = \bigoplus_{h\in H} b_h \bigoplus_{i=1}^{m} \bigoplus_{\substack{g\in\Gamma:\\ \sigma_g(i) = \tau\sigma^{-1}(i)}} c_i^{g} X_g$
- 2 orbits case: $A = \bigoplus_{h_1\in H_1} b_{h_1} \bigoplus_{h_2\in H_2} b_{h_2}$.

5 OUTRO

What next? Testing the ground on...

- How to recover further details on the algebra structure (in these examples and others)?

- A generalization of these two generalizations?

- Extensions of fusion rings: how do NIM-reps keep track of any changes?

- Other exotic fusion rings and their extensions: NIM-reps?

...

Hoping to report more soon!

THANK YOU VERY MUCH FOR YOUR ATTENTION !!!

CONGRATULATIONS AGAIN, SVEN + ARNE! :)

QUESTIONS?

Maybe later?
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