Non-combinatorial involutive braidings: the quantum algebra $\mathfrak{gl}_{k,m}$

Anastasia Doikou

Heriot-Watt University

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Overview

Braidings contain all the (bi) algebraic information.

- Derive new (Hopf)algebra denoted $Y(\mathfrak{gl}_{k,m})$ from a certain non-combinatorial, non-parametric involutive braiding.
- Via the RTT presentation of quantum algebras we derive the $\mathfrak{gl}_{k,m}$ Yangian, which is naturally a Hopf algebra.
- The Yangian $Y(\mathfrak{gl}_{k,m})$ contains an underlying subalgebra, $\mathfrak{gl}_{k,m}$, which itself constitutes a novel non-commutative algebraic structure.
- For $\mathfrak{gl}_{k,m}$: we derive its quadratic relations and defining its Chevalley–Serre type relations –which are distinct from those of the standard general linear Lie superalgebra –and construct Casimir invariants.
- Distinct combinatorial bases for finite-dimensional irreducible representations of the algebra $\mathfrak{gl}_{k,m}$ are also derived.

Overview

- $Y(\mathfrak{gl}_{k,m})$ is distinct from the standard, Yangian of the general linear Lie superalgebra: as a Hopf algebra, $Y(\mathfrak{gl}_{k,m})$ is equipped with the standard, ungraded tensor product algebra structure, such that $(\mathrm{a} \otimes \mathrm{b})(\mathrm{c} \otimes \mathrm{d}) = \mathrm{ac} \otimes \mathrm{bd}$, for $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in Y(\mathfrak{gl}_{k,m})$, avoiding the standard $\mathbb{Z}_2$-graded super-commutativity signs: $(\mathrm{a} \otimes \mathrm{b})(\mathrm{c} \otimes \mathrm{d}) = (-1)^{|b||c|}\mathrm{ac} \otimes \mathrm{bd}$, for $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in Y(\mathfrak{gl}_{k,m})$,

- This requires deforming the algebra coproducts, and introducing a family of additional group elements into the algebra, both of which are dictated by the RTT presentation. This framework generates super-type symmetries through coproduct deformation, leaving the underlying tensor multiplication ungraded.

Preliminaries

- A solution to the Yang-Baxter equation in its braided form is a linear map $\check{r} : V \otimes V \to V \otimes V$ satisfying the relation:

Braid Equation

$$(\check{r} \otimes id_V)(id_V \otimes \check{r})(\check{r} \otimes id_V) = (id_V \otimes \check{r})(\check{r} \otimes id_V)(id_V \otimes \check{r}).$$

If $\check{r}$ is a solution such that $\check{r}^2 = id_{V \otimes V}$, then $\check{r}$ is said to be *involutive*.

- Introduce the linear map $r : V \otimes V \to V \otimes V$, such that $r = \mathrm{P}\check{r}$, where $\mathrm{P} : V \otimes V \to V \otimes V$ is the permutation or flip map: $\mathrm{P}(u \otimes w) = w \otimes u$, $u, w \in V$. Then, r satisfies the YBE

Yang-Baxter Equation

$$r_{12}\ r_{13}\ r_{23} = r_{23}\ r_{13}\ r_{12},$$

If $\check{r}$ is involutive then r satisfies $r_{12} r_{21} = \mathrm{id}_{V \otimes V}$ and is called *reversible*.

Notation

- If $r = \sum\limits_j a_j \otimes b_j$, $a_j, b_j \in \mathrm{End}(V)$, then $r_{12} = \sum\limits_j a_j \otimes b_j \otimes id_V$, $r_{23} = \sum\limits_j id_V \otimes a_j \otimes b_j$ and $r_{13} = \sum_j a_j \otimes id_V \otimes b_j$.
- Let X be a non empty finite set of cardinality n: $\{e_x\}_X$ is the standard canonical basis of $\mathbb{C}^n$, and the $n \times n$ matrices $e_{x,y} := e_x e_y^T$ (^{T} denotes transposition), such that $(e_{x,y})_{z,w} = \delta_{x,z}\delta_{y,w}$, $x, y, z, w \in X$. $\{e_{x,y}\}_{x,y\in X}$ is a basis of $\mathrm{End}(\mathbb{C}^n)$.

Preliminaries

Classes of braidings:

- **Set-theoretic or combinatorial solutions** of the braid (YBE). Let X be a nonempty finite set, for $a, b \in X$, $\sigma_a, \tau_b : X \to X$ are bisections and $r : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$,

$$r(e_a \otimes e_b) = e_{\sigma_a(b)} \otimes \tau_b(a)$$

e.g. the flip map σ and τ are identity maps, i.e. $r(a \otimes b) = b \otimes a$.

By [*AD*] and [*AD, Rybolowicz, Stefanelli*]:

- Involutive ($r^2 = \mathrm{id}$) set-theoretic braiding come for the flip map via a combinatorial Drinfel'd twist.
- Invertible (non-involutive) set-theoretic solutions are coming from racks via the same Drinfel'd twist.
- Set-theoretic solutions $\to$ quantum algebras; requiring them to be Hopf algebras leads naturally to (skew) braces.

Preliminaries

- **"Quantum" or q-deformed solutions** (or doubly deformed) braids ([*Jimbo, Drinfel'd,...*]), $q \in \mathbb{C}$, (quantum groups, Yangian, ...) e.g.:

$$r(e_x \otimes e_y) = \begin{cases} e_x \otimes e_x, & x = y, \\ q^{-1}\ e_y \otimes e_x, & x < y \\ q^{-1}\ e_y \otimes e_x + (1-q^{-2})\ e_x \otimes e_y, & x > y. \end{cases},$$

- **Non-parametric, non-combinatorial solutions** [*Lechner, Pennig & Wood*]), simplest example:

$$r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Baxterization & solutions of YBE

- Let $\check{r} : V \otimes V \to V \otimes V$ solution of the braid equation and $\check{r}^2 = \mathrm{id}_{n^2}$ (involutive).
- Given that $\check{r}$ satisfies the braid & involution, *Baxterization* [*Jones*]:

Baxterized solution

$$\check{R}(\lambda) = \lambda \check{r} + \mathrm{id} \;\Rightarrow\; R(\lambda) = \lambda \mathbf{r} + \mathrm{P}, \quad \text{where} \quad r = \mathrm{P}\check{r}$$

- In the special case $\check{r} = \mathrm{P} = \sum_{x,y \in X} e_{x,y} \otimes e_{y,x}$ $(r = 1_{n^2})$ we recover the **Yangian**.
- Given any involutive $\check{r}$ via Baxterization and QISM, obtain hierarchy of commuting Hams (quantum integrability). Also, quantum groups (Hopf algebras) emerge naturally.

No-combinatorial, non parametric braiding

- We examine a special class of involutive solutions as non-parametric diagonal deformations of the permutation map.
- **Notation.** The set of all non-negative integers: $\mathbb{N}$. The set of all positive integers: $\mathbb{N}^*$. $[n] := \{1, 2, \dots, n\}$ and $X_k^+ := \{k+1, k+2, \dots, n\}$, $k < n$.

Proposition

Let $\mathrm{P} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ be the permutation map explicitly expressed as $\mathrm{P} = \sum_{x,y \in [n]} e_{x,y} \otimes e_{y,x}$. Let also $\mathrm{D} = id - 2\alpha \sum_{x \in X_k^+} e_{x,x} \otimes e_{x,x}$, then $\check{r} = \mathrm{DP}$ is an involutive solution of the braid equation if and only if $\alpha = 0$ or $\alpha = 1$.

Non-combinatorial braidings

1. Let $B = \{e_x\}_{x\in[n]}$ be the standard canonical basis of $\mathbb{C}^n$, then the action of the $\check{r}$-matrix of on $B^{\otimes 2}$ is given by ($\alpha \in \{0,1\}$)

$$\check{r}(e_x \otimes e_y) = \begin{cases} e_x \otimes e_x, & x = y \in [k], \\ (1-2\alpha)e_x \otimes e_x, & x = y \in X_k^+, \\ e_y \otimes e_x, & x \neq y \in [n]. \end{cases}$$

   Notice that if $\alpha = 0$ one recovers the flip map (combinatorial solution), whereas if $\alpha = 1$ the solution is non-combinatorial.

Next: quadratic algebras $\mathfrak{A}$ associated with solutions to the Yang–Baxter equation (FRT construction, and consequence of quasi-triangular Hopf algebras).

The FRT formulation or RTT presentation

[*Faddeev, Reshetikhin, Takhtajan*]

Definition

Let $R(\lambda) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be a solution of the Yang-Baxter equation, $\lambda \in \mathbb{C}$. Let also $L(\lambda) := \sum_{x,y\in[n]} e_{x,y} \otimes L_{x,y}(\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{A}$, $L_{x,y}(\lambda) = \sum_{p\in\mathbb{N}} \lambda^{-p} L^{(p)}_{x,y} \in \mathfrak{A}$, where $\mathfrak{A}$ is the quantum algebra associated to the solution R, and is defined as the quotient of the free unital, associative $\mathbb{C}$-algebra, generated by indeterminates $\left\{ L^{(p)}_{x,y} |\ x, y \in [n],\ p \in \mathbb{N} \right\}$ and relations

$$R_{12}(\lambda_1, \lambda_2)\ L_1(\lambda_1)\ L_2(\lambda_2) = L_2(\lambda_2)\ L_1(\lambda_1)\ R_{12}(\lambda_1, \lambda_2),$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$, $R_{12} = R \otimes 1_{\mathfrak{A}}$, $L_1 = \sum_{x,y\in[n]} e_{x,y} \otimes id_n \otimes L_{x,y}$,

$L_2 = \sum_{x,y\in[n]} id_n \otimes e_{x,y} \otimes L_{x,y}$ (id_n denotes the $n \times n$ identity matrix).

- In L in addition to the indices 1 and 2 there is also an implicit "quantum index" 3 associated to $\mathfrak{A}$, i.e. one writes $L_{13},\ L_{23}$. We retain ungraded tensor products, i.e. $(\mathrm{a} \otimes \mathrm{b})(\mathrm{c} \otimes \mathrm{d}) = \mathrm{ac} \otimes \mathrm{bd}$, for $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in \mathfrak{A}$.
- Vertex algebras as central extensions of the RTT presentation, via a twist [*Kulish, Reshetihkin, Koroshkin, ...*]

The Yangian $Y(\mathfrak{gl}_{k,m})$

Proposition. Let $R(\lambda) = \lambda r + \mathrm{P}$ (P is the permutation map) a solution of the YBE. Consider $r = id - 2\sum_{x \in X_k^+} e_{x,x} \otimes e_{x,x}$, $k + m = n$. Also, consider $L_{x,y}^{(0)} = \delta_{x,y} h_x$, $h_x = 1$ for all $x \in [k]$, h_x is an invertible element with $h_x \neq 1$, for all $x \in X_k^+$. Then, the associated quantum algebra, denoted $Y(\mathfrak{gl}_{k,m})$, is generated by indeterminates $L_{x,y}^{(p)}$, $x, y \in [n]$, $p \in \mathbb{N}$ and relations for all $x, y, z \in [n]$, $l, p \in \mathbb{N}$:

$$\begin{aligned}
&\left[h_x,\ h_y\right] = 0, x, y \in X_k^+,\\
&\left[L_{x,y}^{(p)}, h_z\right] = 2L_{x,y}^{(p)} h_x \delta_{z,x} - 2h_y L_{x,y}^{(p)} \delta_{z,y}, \quad z \in X_k^+\\
&\left[L_{x,y}^{(p+1)},\ L_{z,w}^{(l)}\right] - \left[L_{x,y}^{(p)},\ L_{z,w}^{(l+1)}\right] - 2\left(L_{x,y}^{(p+1)} L_{x,w}^{(l)} - L_{x,y}^{(p)} L_{x,w}^{(l+1)}\right)\delta_{x,z}\big|_{x \in X_k^+}\\
&\qquad +2\left(L_{z,y}^{(l)} L_{x,y}^{(p+1)} - L_{z,y}^{(l+1)} L_{x,y}^{(p)}\right)\delta_{y,w}\big|_{y \in X_k^+} = L_{z,y}^{(l)} L_{x,w}^{(p)} - L_{z,y}^{(p)} L_{x,w}^{(l)}
\end{aligned}$$

where $[\ ,\ ] : Y(\mathfrak{gl}_{k,m}) \times Y(\mathfrak{gl}_{k,m}) \to Y(\mathfrak{gl}_{k,m})$, such that $[\mathrm{a},\ \mathrm{b}] = \mathrm{ab} - \mathrm{ba}$, for all $\mathrm{a}, \mathrm{b} \in Y(\mathfrak{gl}_{k,m})$.

The algebra $\mathfrak{gl}_{k,m}$

Corollary

The algebra $\mathfrak{gl}_{k,m}$ generated by indeterminates h_z, $z \in X_k^+$, $L_{x,y}^{(1)}$, $x, y \in [n]$ and relations,

$$\big[h_x,\ h_y\big] = 0, \quad x, y \in X_k^+$$
$$\big[L_{x,y}^{(1)},\ h_z\big] = 2L_{x,y}^{(1)} h_x \delta_{z,x} - 2h_y L_{x,y}^{(1)} \delta_{y,z}, \quad z \in X_k^+$$
$$[L_{x,y}^{(1)},\ L_{z,w}^{(1)}] - 2L_{x,y}^{(1)} L_{x,w}^{(1)} \delta_{x,z}\big|_{x \in X_k^+} + 2L_{z,y}^{(1)} L_{x,y}^{(1)} \delta_{y,w}\big|_{y \in X_k^+} = L_{z,y}^{(1)} h_x \delta_{x,w} - h_y L_{x,w}^{(1)} \delta_{y,z}$$

is a subalgebra of the Yangian $Y(\mathfrak{gl}_{k,m})$.
Note in particular $(L_{x,y}^{(1)})^2 = 0$, if $x \in [k]$ and $y \in X_k^+$, or $x \in X_k^+$ and $y \in [k]$.

- The nilpotency condition $(L_{x,y}^{(1)})^2 = 0$ serves as hallmark of supersymmetry. It allows $\mathfrak{gl}_{k,m}$ to function as a superalgebra despite being defined in a different framework that bypasses $\mathbb{Z}_2$-gradation.

- **Remark.** There is a map $Y(\mathfrak{gl}_{k,m}) \to \mathfrak{gl}_{k,m}$, such that $L^{(p)}_{x,y} \mapsto 0$, for $p \geq 2$, i.e. $L_{x,y}(\lambda) \mapsto \delta_{x,y} h_x + \lambda^{-1} L^{(1)}_{x,y}$ and $L(\lambda) = L^{(0)} + \lambda^{-1} L^{(1)}$.
- **Remark.** Let $L^{(1)}_{x,x+1} =: e_x$, $L^{(1)}_{x+1,x} =: f_x$, $L^{(1)}_{x,x} =: \epsilon_x$ (Chevalley-Serre generators) Then every $L^{(1)}_{x,y}$, $|x-y| > 1$ is derived by iteration via $\big[L^{(1)}_{x,y},\ L^{(1)}_{y,z}\big] = -h_y L^{(1)}_{x,z}$ for $x < y < z$ (or $x > y > z$).
- **Example.** We consider the simple example of the algebra $\mathfrak{gl}_{1,1}$, generated by elements $h := h_2,\ e := e_1,\ f := f_1,\ \epsilon_1,\ \epsilon_2$ and relations,

$$\begin{gathered}
\big[\epsilon_1,\ \epsilon_2\big] = \big[h, \epsilon_j\big] = 0, \quad j \in [2], \qquad \big[\epsilon_1,\ e\big] = -e, \qquad \big[\epsilon_1,\ f\big] = f \\
\big[\big[\epsilon_2,\ e\big]\big] = eh, \qquad \big[\big[\epsilon_2, f\big]\big] = -hf, \qquad \big[\big[f,\ h\big]\big] = 0, \qquad \big[\big[e,\ h\big]\big] = 0 \\
\big[f,\ e\big] = \epsilon_1 h - \epsilon_2, \qquad e^2 = f^2 = 0,
\end{gathered}$$

where $[[,]] : \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathfrak{gl}_n$, such that $[[a, b]] = ab + ba$.

The algebra $\mathfrak{gl}_{k,m}$

- If $h_x^2 = 1$ for all $x \in [n]$, and we set $\hat{e}_x := h_x e_x$, $\hat{f}_x := h_{x+1} f_x$, $x \in [n-1]$ ($h_x = 1$, if $x \in [k]$) and $\hat{\epsilon}_x = \epsilon_x$ if $x \in [k]$, $\hat{\epsilon}_x = -h_x \epsilon_x$ if $x \in X_k^+$, then:

$$\big[\big[\hat{\xi}_x,\ h_x\big]\big] = \big[\big[\hat{\xi}_x,\ h_{x+1}\big]\big] = 0,\ \xi_x \in \big\{\hat{e}_x,\ \hat{f}_x,\big\},\quad x,\ x+1 \in X_k^+$$

$$\begin{aligned}
&\big[\hat{\epsilon}_x,\ \hat{e}_x\big] = -\hat{e}_x,\quad \big[\hat{\epsilon}_x,\ \hat{f}_x\big] = \hat{f}_x,\quad x \in [n-1],\\
&\big[\hat{\epsilon}_{x+1},\ \hat{e}_x\big] = \hat{e}_x,\ \big[\hat{\epsilon}_{x+1},\ \hat{f}_x\big] = -\hat{f}_x,\quad x \in [n-1],\\
&\big[\hat{f}_x,\ \hat{e}_x\big] = \hat{\epsilon}_x - \hat{\epsilon}_{x+1},\quad k \neq x \in [n],\quad \big[\big[\hat{f}_k,\ \hat{e}_k\big]\big] = \hat{\epsilon}_k + \hat{\epsilon}_{k+1}.
\end{aligned}$$

and

$$\big[\big[\hat{f}_x, \hat{e}_{x+1}\big]\big] = 0,\quad x+1 \in X_k^+,\quad \big[\big[\hat{f}_x, \hat{e}_{x-1}\big]\big] = 0,\quad x \in X_k^+,$$

Also, $\hat{e}_k^2 = \hat{f}_k^2 = 0$, and for all $k \neq x \in [n]$, $\hat{\xi}_x \in \big\{\hat{e}_x,\ \hat{f}_x\big\}$ the cubic relations become,

$$\hat{\xi}_{x\pm1}\ \hat{\xi}_x^2 + \hat{\xi}_x^2\ \hat{\xi}_{x\pm1} - 2c_x\hat{\xi}_x\ \hat{\xi}_{x\pm1}\ \hat{\xi}_x = 0,\qquad c_x = \begin{cases} 1, & \text{if}\quad x \in [k-1],\\ -1, & \text{if}\quad x \in X_k^+. \end{cases}$$

1. We observe that the main quadratic relations among the elements $\hat{\epsilon}_x$, $\hat{e}_x$, and $\hat{f}_x$ are identical to those of the general linear Lie superalgebra $\mathfrak{gl}(k|m)$, however extra quadratic anti-commutation relations exist. The cubic relations are also slightly modified relative to $\mathfrak{gl}(k|m)$; specifically, $c_x = -1$ if $x \in X_k^+$.

2. Note that for the algebra $\mathfrak{gl}_{1,1}$, the elements $\hat{e}_1$, $\hat{f}_1$, $\hat{\epsilon}_1$, and $\hat{\epsilon}_2$ correspond precisely to the Chevalley–Serre generators of the standard algebra $\mathfrak{gl}(1|1)$.

The Hopf algebra

Proposition

Let $Y(\mathfrak{gl}_{k,m})$ be the unital, associative algebra derived earlier. Then $(Y(\mathfrak{gl}_{k,m}), \Delta, \epsilon, s)$ is a Hopf algebra with the following maps:

1. a coproduct $\Delta : Y(\mathfrak{gl}_{k,m}) \to Y(\mathfrak{gl}_{k,m}) \otimes Y(\mathfrak{gl}_{k,m})$, such that $\Delta(h_x) = h_x \otimes h_x$, for all $x \in X_k^+$ and $\Delta(L_{x,y}^{(p)}) = \sum_{z\in[n]} \sum_{\substack{p_1+p_2=p \\ p_1,p_2\in\mathbb{N}}} L_{z,y}^{(p_1)} \otimes L_{x,z}^{(p_2)}$, for all $x, y \in [n]$.
2. a counit $\epsilon : Y(\mathfrak{gl}_{k,m}) \to \mathbb{C}$, such that $\epsilon(h_x) = 1$, for all $x \in X_k^+$ and $\epsilon(L_{x,y}^{(p)}) = 0$, for all $x, y \in [n]$, $p \in \mathbb{N}^*$.
3. an antipode $s : Y(\mathfrak{gl}_{k,m}) \to Y(\mathfrak{gl}_{k,m})$, such that $s(h_x) = h_x^{-1}$, for all $x \in X_k^+$ and
$$\sum_{z\in[n]} \sum_{\substack{p_1+p_2=p \\ p_1,p_2\in\mathbb{N}}} s(L_{z,y}^{(p_1)})L_{x,z}^{(p_2)} = \sum_{z\in[n]} \sum_{\substack{p_1+p_2=p \\ p_1,p_2\in\mathbb{N}}} L_{z,y}^{(p_1)} s(L_{x,z}^{(p_2)}) = 0.$$

The Hopf algebra

For $\mathfrak{gl}_{k,m}$ in particular:

- For all $x, y \in [n]$,
  $\Delta(h_x) = h_x \otimes h_x, \quad x \in X_k^+, \ \ \Delta(L_{x,y}^{(1)}) = L_{x,y}^{(1)} \otimes h_x + h_y \otimes L_{x,y}^{(1)},$
- The counit for all $x, y \in [n]$, $\epsilon(h_x) = 1$, $\epsilon(L_{x,y}^{(1)}) = 0$.
- The antipode, for all $x, y \in [n]$,
  $s(h_x) = h_x^{-1}, \ \ s(L_{x,y}^{(1)}) = -h_y^{-1} L_{x,y}^{(1)} h_x^{-1}$
- If $h_x^2 = 1$, and set $\hat{L}_{x,y} := h_x L_{x,y}$ then:

$$\Delta(\hat{L}_{x,y}) = L_{x,y} \otimes 1 + h_x h_y \otimes \hat{L}_{x,y}.$$

In particular,

$$\Delta(\hat{\xi}_x) = \hat{\xi}_x \otimes 1 + h_x h_{x+1} \otimes \hat{\xi}_x, \quad \hat{\xi}_x \in \left\{\hat{e}_x, \hat{f}_x\right\}$$
$$\Delta(\hat{\epsilon}_x) = \hat{\epsilon}_x \otimes 1 + 1 \otimes \hat{\epsilon}_x.$$

The fundamental representation

1. The map $\pi : \mathfrak{gl}_{k,m} \to \mathrm{End}(\mathbb{C}^n)$, such that for all $x, y \in [n]$,

$$L^{(1)}_{x,y} \mapsto e_{y,x}, \quad h_x \mapsto id_n, \ \text{ if } \ x \in [k], \quad h_x \mapsto d_x := id_n - 2e_{x,x}, \ \text{ if } \ x \in X^+_k$$

   is an algebra representation (the fundamental representation).

2. Consider $L(\lambda) = L^{(0)} + \lambda^{-1} L^{(1)}$, where $L^{(0)} = \sum_{x \in [n]} e_{x,x} \otimes h_x$, $L^{(1)} = \sum_{x,y \in [n]} e_{x,y} \otimes L^{(1)}_{x,y}$ and $h_x, L^{(1)}_{x,y} \in \mathfrak{gl}_{k,m}$, Then $(id \otimes \pi) L^{(0)} = \mathrm{D}$ and $(id \otimes \pi) L^{(1)} = \mathrm{P}$, where recall $\mathrm{D} = \sum_{x \in [n]} e_{x,x} \otimes d_x = \sum_{x \in [n]} d_x \otimes e_{x,x}$ and $\mathrm{P} = \sum_{x,y \in [n]} e_{x,y} \otimes e_{y,x}$, i.e. $(id \otimes \pi) L(\lambda) = R(\lambda)$.

- In the following proposition we derive central elements of $\mathfrak{gl}_{k,m}$. We first introduce the notion of a trace for elements $A = \sum_{x,y\in[n]} e_{x,y} \otimes A_{x,y} \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{A}$, where $\mathfrak{A}$ is some quantum algebra. Then we define the trace of A, $tr(A) := \sum_{x\in[n]} A_{x,x} \in \mathfrak{A}$.

Proposition

Let $L = L^{(0)} + \lambda^{-1} L^{(1)} \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{gl}_{k,m}$, where $L^{(0)} = \sum_{x\in[n]} e_{x,x} \otimes h_x$ and $L^{(1)} = \sum_{x,y\in[n]} e_{x,y} \otimes L^{(1)}_{x,y}$ such that for all $x, y \in [n]$, $h_x, L^{(1)}_{x,y} \in \mathfrak{gl}_{k,m}$ $(n = k + m)$. Let also $t(\lambda) := L(\lambda)L^{-1}(-\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{gl}_{k,m}$ (Sklyanin's rep.) and $\tau(\lambda) := tr\big(d\ t(\lambda)\big) \in \mathfrak{gl}_{k,m}$, where $d = \sum_{x\in[n]} \theta_x e_{x,x}$ and $\theta_x = \begin{cases} 1, & \text{if } x \in [k] \\ -1, & \text{if } x \in X_k^+ \end{cases}$.

Then, $\big[\tau(\lambda),\ \mathfrak{g}\big] = 0$, $\mathfrak{g} \in \mathfrak{gl}_{k,m}$.

- **Proof.** Recall, $L(\lambda) = L^{(0)} + \lambda^{-1} L^{(1)}$ satisfies the RTT relation with $R(\lambda) = r + \lambda^{-1}\mathrm{P}$, where $r = \mathrm{D}$, then

$$(\pi \otimes id)\Delta^{(op)}(\mathfrak{g})L(\lambda) = L(\lambda)(\pi \otimes id)\Delta(\mathfrak{g}), \quad \mathfrak{g} \in \mathfrak{gl}_{k,m},$$

where $\Delta^{(op)} = \sigma \circ \Delta$, σ is the algebra flip map, $\sigma : Y(\mathfrak{gl}_{k,m}) \otimes Y(\mathfrak{gl}_{k,m}) \to Y(\mathfrak{gl}_{k,m}) \otimes Y(\mathfrak{gl}_{k,m})$, $a \otimes b \mapsto b \otimes a$.Then,

$$(\pi \otimes id)\Delta^{(op)}(\mathfrak{g})t(\lambda) = t(\lambda)(\pi \otimes id)\Delta^{(op)}(\mathfrak{g}).$$

This leads to $\big[L_{x,y},\ \tau(\lambda)\big] = \big[h_x,\ \tau(\lambda)\big] = 0,$ for all $x, y \in [n]$.

- **Example.** The first two non-trivial elements of the expansion $\tau(\lambda) = \sum_{n\geq 0} \lambda^{-n}\tau^{(n)}$ are
$$\tau^{(1)} = 2\sum_{x\in[n]} \theta_x L^{(1)}_{x,x} h_x^{-1}, \qquad \tau^{(2)} = 2\sum_{x,y\in[n]} \theta_x L^{(1)}_{x,y} h_y^{-1} L^{(1)}_{y,x} h_x^{-1}.$$
The elements $\tau^{(1)}$, $\tau^{(2)}$ are the linear and quadratic Casimir elements of $\mathfrak{gl}_{k,m}$ respectively. By setting $\hat{L}_{x,y} := h_x^{-1} L^{(1)}_{x,y}$, the Casimir elements become,
$$\tau^{(1)} = 2\sum_{x\in[n]} \theta_x \hat{L}_{x,x} \quad \textit{and} \quad \tau^{(2)} = 2\sum_{x,y\in[n]} \theta_x \hat{L}_{x,y}\hat{L}_{y,x}.$$

Notation

Next construct combinatorial bases of irreps of $\mathfrak{gl}_{k,m}$

- We introduce the shorthand notation:

Notation

$$\mathcal{E}_x := \pi^{\otimes N}\Delta^{(N)}(\epsilon_x),\ \mathfrak{h}_x := \pi^{\otimes N}\Delta^{(N)}(h_x),$$
$$\mathfrak{e}_x := \pi^{\otimes N}\Delta^{(N)}(e_x),\ \mathfrak{f}_x := \pi^{\otimes N}\Delta^{(N)}(f_x),$$
$$\mathfrak{t}_{x,y} := \pi^{\otimes N}\Delta^{(N)}(L_{x,y})$$

$x, y \in [n]$.

- Recall ϵ_x, $x \in [n]$, e_x, f_x, $x \in [n-1]$ are the Chevalley-Serre type elements and the map $\pi : \mathfrak{gl}_{k,m} \to \mathrm{End}(\mathbb{C}^n)$ is the fundamental rep.

Notation

- Let $\mathfrak{u}^+ := e_1^{\otimes N}$ and $\mathfrak{u}^- := e_n^{\otimes N}$. Notice, $\mathfrak{f}_x \mathfrak{u}^+ = 0,\ \mathfrak{e}_x \mathfrak{u}^- = 0,$ for all $x \in [n-1]$. We also introduce $\mathfrak{u}^{\pm}_{m_1, m_2, \ldots, m_n} \in (\mathbb{C}^n)^{\otimes N}$:

$$\mathfrak{u}^{+}_{m_1, m_2, \ldots, m_n} := \mathfrak{t}_{1,n}^{m_n} \mathfrak{t}_{1,n-1}^{m_{n-1}} \ldots \mathfrak{t}_{1,3}^{m_3} \mathfrak{t}_{1,2}^{m_2}\ \mathfrak{u}^+,$$
$$\mathfrak{u}^{-}_{m_1, m_2, \ldots, m_n} := \mathfrak{t}_{n,n-1}^{m_{n-1}} \mathfrak{t}_{n,n-2}^{m_{n-2}} \ldots \mathfrak{t}_{n,2}^{m_2} \mathfrak{t}_{n,1}^{m_1}\ \mathfrak{u}^-$$

  $\sum_{x \in [n]} m_x = N$. Recall $\big[\mathfrak{t}_{1,x},\ \mathfrak{t}_{1,y}\big] = 0$ and $\big[\big[\mathfrak{t}_{n,x}, \mathfrak{t}_{n,y}\big]\big] = 0$, for all $x, y \in [n]$.

  Due to $L^2_{x,y} = 0$ for $x \in [k]$ and $y \in X_k^+$ or $x \in X_k^+$ and $y \in [k]$:

  1. in $\mathfrak{u}^{+}_{m_1, m_2, \ldots, m_n}$, $m_x \in \big\{0, 1, \ldots, N\big\}$ if $x \in [k]$ and $m_x \in \big\{0, 1\big\}$ if $x \in X_k^+$.
  2. in $\mathfrak{u}^{-}_{m_1, m_2, \ldots, m_n}$, $m_x \in \big\{0, 1, \ldots, N\big\}$ if $x \in X_k^+$ and $m_x \in \big\{0, 1\big\}$ if $x \in [k]$.

Irreducible representations of $\mathfrak{gl}_{k,m}$: combinatorial bases

Theorem

The action of $\mathfrak{gl}_{k,m}$ on $\mathfrak{u}^{\pm}_{m_1,m_2,\ldots,m_n}$, $\sum\limits_{y\in[n]} m_y = N$ is given by,

$$\begin{aligned}
\mathfrak{h}_x\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots} &= (-1)^{m_x}\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots}, \quad x\in X_k^+\\
\mathcal{E}_x\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots} &= m_x\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots}, \quad x\in[k]\\
\mathcal{E}_x\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots} &= m_x\ (-1)^{m_x-1}\mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots}, \quad x\in X_k^+\\
\mathfrak{e}_x\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots} &= b_x^{\pm}\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x-1,m_{x+1}+1,\ldots}, \quad x\in[n-1]\\
\mathfrak{f}_x\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x,m_{x+1},\ldots} &= c_x^{\pm}\ \mathfrak{u}^{\pm}_{m_1,\ldots,m_x+1,m_{x+1}-1,\ldots}, \quad x\in[n-1],
\end{aligned}$$

where

- for all $x\neq 1$, $b_x^+ = m_x$, $c_x^+ = m_{x+1}$, if $x+1\in[k]$;
  $b_x^+ = (-1)^{m_{x+1}} m_x$, $c_x^+ = m_{x+1}$, if $x+1\in X_k^+$,
- $b_1^+ = 1$, $c_1^+ = (m_1+1)m_2$,
- for all $x\neq n-1$, $b_x^- = m_x$, $c_x^- = m_{x+1}$, if $x+1\in[k]$;
  $b_x^- = (-1)^{m_{x+1}+m_x-1} m_x$, $c_x^- = m_{x+1}$, if $x+1\in X_k^+$,
- $b_{n-1}^- = (-1)^{m_n+m_{n-1}-1} m_{n-1}(m_n+1)$, $c_{n-1}^- = 1$.

The sets $\left\{\mathfrak{u}^{\pm}_{m_1,m_2,\ldots,m_n}\right\}_{\sum m_j=N}$ form natural bases for distinct irreducible representations of $\mathfrak{gl}_{k,m}$. We refer to these as *combinatorial bases*.

Mutual centralizers & spectrum of Hams

- We next investigate the symmetry of involutive tensor representations of the braid groups $\to$ spectrum of spin chain like Hamiltonians.

Proposition

Let $\rho : B_N \to \mathrm{End}((\mathbb{C}^n)^{\otimes N})$, $\sigma_j \mapsto \check{r}_j$, where $\check{r}_j := id^{\otimes (j-1)} \otimes \check{r} \otimes id^{\otimes (N-j-1)}$ and $\check{r} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is the involutive solution of the braid equation given earlier. Let also $\pi : \mathfrak{gl}_{k,m} \to \mathrm{End}(\mathbb{C}^n)$ be the fundamental representation of $\mathfrak{gl}_{k,m}$. Then

$$\big[\check{r}_j,\ \pi^{\otimes N}\Delta^{(N)}(\mathfrak{g})\big] = 0, \quad \forall\ \mathfrak{g} \in \mathfrak{gl}_{k,m}, \quad j \in [N-1].$$

The proof is based on the Yang-Baxter equation and the form of the R-matrix.

- Consider also the sum of all the generators of the braid group B_N, $\mathcal{H} := \sum_{j\in[N-1]} \check{r}_j \in \mathrm{End}((\mathbb{C}^n)^{\otimes N})$, where $\check{r}$ is $\mathfrak{gl}_{k,m}$ symmetric. Then:

$$\mathcal{H}\mathfrak{u}^{\pm}_{m_1,\ldots,m_n} = \pm(N-1)\mathfrak{u}^{\pm}_{m_1,\ldots,m_n}.$$

Study of spectral decomposition of $\mathcal{H} \to$ bases for irreps of $\mathfrak{gl}_{k,m}$.

Example: spectral decomposition

- Let $\check{r}$ be the $\mathfrak{gl}_{k,m}$ invariant solution of the braid equation and $\{e_x\}_{x\in[n]}$ be the standard canonical basis of $\mathbb{C}^n$. Then, $\check{r}$ has 2 eigenvalues:

  1. $\lambda_1 = 1$ with multiplicity $\frac{n(n-1)}{2} + k$ and corresponding eigenvectors:
     $u^+_{1,1} := e_1 \otimes e_1,\ \ u^+_{j,j} := \mathfrak{t}^2_{1,j} u^+_{1,1} = 2e_j \otimes e_j,\ j \in [k],$
     $u^+_{1,j} := \mathfrak{t}_{1,j} u^+_{1,1} = e_1 \otimes e_j + e_j \otimes e_1,\ j \in \{2, 3, \dots, n\},$
     $u^+_{i,j} := \mathfrak{t}_{1,j}\mathfrak{t}_{1,i} u^+_{1,1} = e_i \otimes e_j + e_j \otimes e_i,\ i < j \in \{2, 3, \dots, n\}.$

  2. $\lambda_2 = -1$ with multiplicity $\frac{n(n+1)}{2} - k$ and corresponding eigenvectors:
     $u^-_{n,n} = e_n \otimes e_n,\ \ u^-_{j,j} = \mathfrak{t}^2_{n,j} u^-_{n,n} = 2e_j \otimes e_j,\ \ j \in X^+_k,$
     $u^-_{n,j} := \mathfrak{t}_{n,j} u^-_{n,n} = e_n \otimes e_j - e_j \otimes e_n,\ \ j \in [n-1],$
     $u^-_{i,j} := \mathfrak{t}_{n,j}\mathfrak{t}_{n,i} u^-_{n,n} = e_j \otimes e_i - e_i \otimes e_j,\ \ i < j \in [n-1].$

Example: spectral decomposition

- That is,

$$V_n^{\otimes 2} = V_{\lambda_1} \oplus V_{\lambda_2}$$

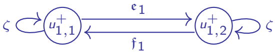
$$\dim V_n = n, \;\; \dim V_{\lambda_1} = \tfrac{n(n-1)}{2} + k, \;\; \dim V_{\lambda_2} = \tfrac{n(n+1)}{2} - k.$$

Each eigenspace is invariant under the action of $\mathfrak{gl}_{k,m}$ and the eigenvectors provide bases of irreducible representations of $\mathfrak{gl}_{k,m}$ of dimensions $d_1 = \frac{n(n-1)}{2} + k$, $d_2 = \frac{n(n+1)}{2} - k$.

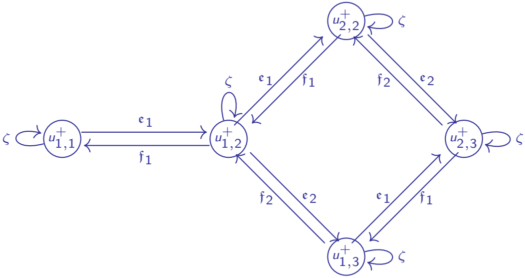
Examples of transitions between basis elements

The action of $\mathfrak{gl}_{k,m}$ on the basis $\{u^+_{x,y}\}$, $x, y \in [n]$, is illustrated for $\mathfrak{gl}_{1,1}$ and $\mathfrak{gl}_{2,1}$. The diagrams depict the transitions between the elements of the basis.

1. $\mathfrak{gl}_{1,1}$, a two-dimensional basis: $\zeta \in \{\mathcal{E}_1, \mathcal{E}_2, \mathfrak{h}\}$



2. $\mathfrak{gl}_{2,1}$, a five-dimensional basis: $\zeta \in \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathfrak{h}_1, \mathfrak{h}_2\}$



Examples of transitions between basis elements

The action of $\mathfrak{gl}_{k,m}$ on the basis $\left\{u^-_{x,y}\right\}$, $x, y \in [n]$, is also depicted schematically for the special cases $\mathfrak{gl}_{1,1}$ and $\mathfrak{gl}_{2,1}$:

1. $\mathfrak{gl}_{1,1}$, a two-dimensional basis: $\zeta \in \left\{\mathcal{E}_1, \mathcal{E}_2, \mathfrak{h}\right\}$

$$\begin{tikzcd} u^-_{2,2} \arrow[loop left, "\zeta"] \arrow[r, shift left, "\mathfrak{f}_1"] & u^-_{1,2} \arrow[loop right, "\zeta"] \arrow[l, shift left, "\mathfrak{e}_1"] \end{tikzcd}$$

2. $\mathfrak{gl}_{2,1}$, a four-dimensional basis: $\zeta \in \left\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathfrak{h}_1, \mathfrak{h}_2\right\}$

$$\begin{tikzcd} u^-_{3,3} \arrow[loop left, "\zeta"] \arrow[r, shift left, "\mathfrak{f}_2"] & u^-_{2,3} \arrow[loop above, "\zeta"] \arrow[l, shift left, "\mathfrak{e}_2"] \arrow[r, shift left, "\mathfrak{f}_1"] & u^-_{1,3} \arrow[loop above, "\zeta"] \arrow[l, shift left, "\mathfrak{e}_1"] \arrow[r, shift left, "\mathfrak{f}_2"] & u^-_{1,2} \arrow[loop right, "\zeta"] \arrow[l, shift left, "\mathfrak{e}_2"] \end{tikzcd}$$

Investigating the spectral decomposition of the $\mathfrak{gl}_{k,m}$-invariant spin-chain Hamiltonian for $N > 2$ is a natural application of the present findings.

The algebra $\mathfrak{gl}_{1,1}$: irreducible representations

- For the $\mathfrak{gl}_{1,1}$-invariant $\check{r}$-matrix the sum $\mathcal{H} = \sum_{j\in[N-1]} \check{r}_j$ corresponds to the Heisenberg XX model in the presence of an external magnetic field:

$$\mathcal{H} = \frac{1}{2}\sum_{j\in[N-1]} \left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + 2\sigma_j^z\right) - \frac{1}{2}\left(\sigma_1^z - \sigma_N^z\right) \in \mathrm{End}((\mathbb{C}^2)^{\otimes N}).$$

Lemma (Highest-weight representations)

Let $\varphi : \mathfrak{gl}_{1,1} \to \mathrm{End}(V)$, such that

$$\epsilon_j \mapsto \varepsilon_j, \quad j \in \{1,2\}, \quad h \mapsto \mathrm{h}, \quad e \mapsto \mathrm{e}, \quad f \mapsto \mathrm{f}.$$

If for some $u \in V$: $\varepsilon_j u = \lambda_j u$, $\mathrm{h}u = \xi u$, $\mathrm{f}u = 0$, λ_j, $\xi \in \mathbb{C}$, $j \in \{1,2\}$ and $w := \mathrm{e}u$, then

$$\mathrm{e}w = 0, \quad \mathrm{h}w = -\xi w, \quad \varepsilon_1 w = (\lambda_1 - 1)w, \quad \varepsilon_2 w = -(\lambda_2 - \xi)w, \quad \mathrm{f}w = (\lambda_1\xi - \lambda_2)u.$$

The proof follows directly from the $\mathfrak{gl}_{1,1}$ algebraic relations.

- *The non-trivial highest-weight representations of* $\mathfrak{gl}_{1,1}$ *are two-dimensional;* $\{u,\ w\}$, *where* $w = \mathfrak{e}u$, *is the basis of the corresponding two-dimensional vector space.* $\zeta \in \{\varepsilon_1, \varepsilon_2, \mathrm{h}\}$

$$\begin{tikzcd} u \arrow[r, shift left, "\mathrm{e}"] \arrow[loop left, "\zeta"] & w \arrow[l, shift left, "\mathrm{f}"] \arrow[loop right, "\zeta"] \end{tikzcd}$$

- We next introduce combinatorial bases of irreducible representations of $\mathfrak{gl}_{1,1}$.

Notation

We denote by $\varpi_p^{(N,p)} \in (\mathbb{C}^2)^{\otimes N}$, $p \in \{0, 1, \ldots, N-1\}$ a linear combination of all possible permutations of $\underbrace{e_1 \otimes \ldots \otimes e_1}_{N-p} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{p}$, such that $\mathfrak{f}\varpi_p^{(N,p)} = 0$.

Proposition (Combinatorial bases)

Let $\varpi_p^{(N,p)} \in (\mathbb{C}^2)^{\otimes N}$, $p \in \{0, 1, \dots, N-1\}$ be such that: $\mathfrak{f}\varpi_p^{(N,p)} = 0$ (recall $\mathfrak{e}, \mathfrak{f}, \mathfrak{h}, \mathcal{E}_1, \mathcal{E}_2$ defined earlier). Let also $\varpi_{p+1}^{(N,p)} := \mathfrak{e}\varpi_p^{(N,p)}$, then:

1. $\mathcal{E}_1\varpi_l^{(N,p)} = (N-l)\varpi_l^{(N,p)}$, $\mathcal{E}_2\varpi_l^{(N,p)} = l(-1)^{l-1}\varpi_l^{(N,p)}$, $\mathfrak{h}\ \varpi_l^{(N,p)} = (-1)^l\varpi_l^{(N,p)}$, $l \in \{p,\ p+1\}$, $\mathfrak{e}\varpi_{p+1}^{(N,p)} = 0$, $\mathfrak{f}\varpi_{p+1}^{(N,p)} = (-1)^p N\varpi_p^{(N,p)}$.
2. $\varpi_{p+1}^{(N,p)} \perp \varpi_{p+1}^{(N,p+1)}$.

Proof.

1. Equ. in part (1) follow from the structure of the states $\varpi_l^{(N,p)}$, $l \in \{p,\ p+1\}$; moreover, $\mathfrak{e}\varpi_{p+1}^{(N,p)} = 0$, due to $\mathfrak{e}^2 = 0$. The last equation is proven via the $\mathfrak{gl}_{1,1}$ relations, in particular $[\mathfrak{f},\ \mathfrak{e}] = \mathcal{E}_1\mathfrak{h} - \mathcal{E}_2$ and $\mathfrak{f}\varpi_p^{(N,p)} = 0$, $\varpi_{p+1}^{(N,p)} = \mathfrak{e}\varpi_p^{(N,p)}$.
2. Recall the inner product in $(\mathbb{C}^2)^{\otimes N}$, i.e. for every $a,\ b \in (\mathbb{C}^2)^{\otimes N}$ the inner product is $\langle a,\ b\rangle := a^\dagger \cdot b$, where †, denotes complex conjugation and transposition. Also, $\mathfrak{e}^T = \mathfrak{h}\mathfrak{f}$ ($\mathfrak{e}^T = \mathfrak{e}^\dagger$). Then,

$$\langle\varpi_{p+1}^{(N,p)},\ \varpi_{p+1}^{(N,p+1)}\rangle = \langle\mathfrak{e}\ \varpi_p^{(N,p)},\ \varpi_{p+1}^{(N,p+1)}\rangle = \langle\varpi_p^{(N,p)},\ \mathfrak{h}\mathfrak{f}\ \varpi_{p+1}^{(N,p+1)}\rangle = 0.$$

Proposition

Let V_{N_i,p_i} be the two-dimensional vector space with a basis $\{\varpi_{p_i}^{(N_i,p_i)},\ \varpi_{p_i+1}^{(N_i,p_i)}\}$, $p_i \in \{0,1,\dots,N_i-1\}$, $i \in \{1,\ 2\}$, as derived earlier. Then,

$$V_{N_1,p_1} \otimes V_{N_2,p_2} = V_{N,p} \oplus V_{N,p+1},$$

where $N = N_1 + N_2$ and $p = p_1 + p_2$.

Proof. Let,

(a) $\varpi_p^{(N,p)} := \varpi_{p_1}^{(N_1,p_1)} \otimes \varpi_{p_2}^{(N_2,p_2)}$ and $\varpi_{p+1}^{(N,p)} := \mathfrak{e}\varpi_p^{(N,p)}$.

(b) $\varpi_{p+1}^{(N,p+1)} := \varpi_{p_1+1}^{(N_1,p_1)} \otimes \varpi_{p_2}^{(N_2,p_2)} - \frac{N_1}{N_2}(-1)^{p_1}\varpi_{p_1}^{(N_1,p_1)} \otimes \varpi_{p_2+1}^{(N_2,p_2)}$ and $\varpi_{p+2}^{(N,p+1)} := \mathfrak{e}\varpi_{p+1}^{(N,p+1)}$.

We then show that $\mathfrak{f}\varpi_q^{(N,q)} = 0, \quad q \in \{p,\ p+1\}$.
Also, it follows that relations of part (1) of the previous Prop. also hold for $\varpi_p^{(N,p)},\ \varpi_{p+1}^{(N,p)}$ and $\varpi_{p+1}^{(N,p+1)},\ \varpi_{p+2}^{(N,p+1)}$ defined in (a) and (b) above.

That is, $V_{N_1,p_1} \otimes V_{N_2,p_2} = V_{N,p} \oplus V_{N,p+1}$, where each $V_{N,q}$, $q \in \{p, p+1\}$ is a two-dimensional vector space with a basis $\{\varpi_q^{(N,q)},\ \varpi_{q+1}^{(N,q)}\}$.

Combinatorial bases and Young tableaux

- We denote $\lambda \vdash N$ a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ of the positive integer N, where λ_i are weakly decreasing positive integers and $\sum_{i \in [p]} \lambda_i = N$. The size of λ is denoted $|\lambda|$, and in general $|\lambda| = N$.

Definition(Young diagram)

Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$, $\lambda \vdash N$ where $p \geq 1$. The Young (or Ferrers) diagram of shape λ is an array of N squares having p rows with row i containing λ_i squares.

Definition

A filling (or weight) of a Young diagram is any way of putting a positive integer in each box of the diagram. Let $\mu = (\mu_1, \mu_2, ..., \mu_l)$ be a filling of a Young diagram. Each μ_i is the number of times the integer i appears in the diagram.

Standard Young tableaux

- In order for the diagram to be completely filled, it is necessary for $|\lambda| = |\mu|$. It is possible to fill diagrams arbitrarily; we impose certain restrictions on the filling μ. These lead to the definition of a Young tableau.

Definition (Standard Young tableaux)

Suppose $\lambda \vdash N$. A Young tableau T is obtained by filling in the boxes of the Young diagram with symbols taken from some alphabet, which is usually required to be a totally ordered set. A Young tableau of shape λ is called a λ-tableau. A Young tableau is standard if the rows and columns of T are increasing sequences. That is, T is filled with the numbers $1, 2, \ldots, N$ bijectively.

- We consider the set $[n]$ with the standard ordering $1 < 2 < \ldots k < k+1 < \ldots < n$. There are various definitions for semi-standard Young tableaux depending on the variation of the associated Schur functions [*Macdonald*].

SSYT

- We use here the following definition of a semi-standard Young tableau.

SSYT

A Young tableau is semi-standard if the filling is:

1. weakly increasing across each row and strictly increasing down each column for numbers $\{1, 2, \ldots, k\}$
2. strictly increasing across each row and weakly increasing down each column for numbers $\{k+1, k+2, \ldots, n\}$.

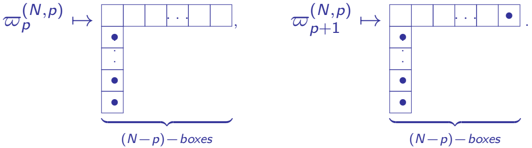
- The *SSYT* defined above are associated to the hook Schur functions also known as super-symmetric Schur functions and correspond to representations of the Lie superalgebra $\mathfrak{gl}(k|m)$. The hook Schur functions [*Berele and Regev*] correspond to the sixth variant of Schur functions considered in [*Macdonald*].

SSYT and vector spaces

- We focus on the algebra $\mathfrak{gl}_{1,1}$. In this case the Young tableaux are filled by 1 and 2, and according to the SSYT Definition the only allowed *SSYTs* are of shape $\lambda = (N-p, \underbrace{1, 1, \dots, 1}_{p})$, with two possible fillings: $\mu_1 = (N-p,\ p)$ and $\mu_2 = (N-p-1,\ p+1)$. The two-element set of *SSYTs* of shape $\lambda = (N-p, \underbrace{1, 1, \dots, 1}_{p})$ is denoted $SSYT(N,p)$.
- Each one of the two elements in $SSYT(N,p)$ corresponds to an element of the basis $\mathrm{B}_{N,p} = \{\varpi_p^{(N,p)},\ \varpi_{p+1}^{(N,p)}\}$ of the two-dimensional vector space $V_{N,p}$. That is, there is a bijective map between the sets $\{\mathrm{B}_{N,p}\}$ and $\{SSYT(N,p)\}$, $p \in \{0, 1, \dots, N-1\}$, such that $\mathrm{B}_{N,p} \mapsto SSYT(N,p)$.

SSYT and vector spaces

Specifically,



- The Young tableaux above correspond to the fillings $\mu_1 = (N-p,\ p)$ and $\mu_2 = (N-p-1,\ p+1)$ respectively. The empty boxes in the tableaux are occupied by 1, whereas the boxes containing a bullet are occupied by 2.